DIFFERENTIAL GEOMETRY OF THE COMPLEX PLANE*

BY

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1. Threads and chains

We shall mean by the *complex plane* that where x and y take complex as well as real values. In counting parameters, we shall always count the number of *real* variables involved. Thus, the plane itself is a *four*-parameter system, in which there are included *one*, *two* and *three*-parameter systems. It is assumed also that all of the functions of real parameters involved are analytic in such regions as may interest us. Finally, when we speak of a system as depending on a certain number of parameters, we mean that it depends upon that number and no less. Thus, we should not include a single point as a special case of a *one*-parameter system. The conjugate of any number a shall be written $\overline{a}$.

A system of points depending analytically upon a single real parameter shall be called a "thread." It is conveniently represented by equations

$$x = x(u), \quad y = y(u), \quad \overline{x} = \overline{x}(u), \quad \overline{y} = \overline{y}(u), \quad u = \overline{u}. \tag{1}$$

Every thread lies upon a curve obtained by allowing u to take all complex values. Two curves can not intersect in a thread, for if irreducible, they have only discreet points in common, and if reducible, they have in common either discreet points, or a common branch. Another way to indicate a thread analytically is in terms of its projections on the two axes:

$$f(x, \overline{x}) = \phi(y, \overline{y}) = 0. \tag{2}$$

There is one thread which is of fundamental importance in all complex geometry, namely, the "chain." This is such a system of collinear points that the cross ratio of any four is real, while the system contains the point which makes any preassigned real cross ratio with three of its members. The real domain

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of a real line is the standard example of a chain, and if the totality of points of a real line be represented in the Gauss plane, the points of a chain will appear as collinear or concyclic points. Two points may be connected by a simply infinite system of chains. Thus, an arbitrary chain connecting (x'', y'') and (x', y') may be written

$$(3) \qquad x = \frac{x''+t\rho x'}{1+t\rho}, \quad y = \frac{y''+t\rho y'}{1+t\rho}, \quad \bar{x} = \frac{\bar{x}''+t\bar{\rho}\,\bar{x}'}{1+t\bar{\rho}}, \quad \bar{y} = \frac{\bar{y}''+t\bar{\rho}\,\bar{y}'}{1+t\bar{\rho}}.$$

Here t is the parameter that changes along the chain, while ρ serves to determine which of the chains connecting the two points we have to deal with.

There is associated with each chain an involutory transformation of its line. Each point of the chain is invariant in this transformation. As for other points of the line, the relation is such that corresponding points are harmonically separated by pairs of points of the chain. In the Gauss plane this appears as a reflection in a line, or inversion in a circle. If two chains be so related to one another that the first is transformed into itself by the transformation associated with the second, then the second is anallagmatic in the transformation associated with the first. We shall call two such chains "orthogonal." They will appear in the Gauss plane as mutually perpendicular circles or lines.

2. Congruences and curves

A system of points which depends upon two parameters shall be called a "congruence." The real domain furnishes one example of such a variety; a curve furnishes another such. The fundamental question connected with any congruence is this: is it, or is it not, a curve? Let the congruence be represented by the equations

$$(4) \quad x = x(u, v), \quad y = y(u, v), \quad \bar{x} = \bar{x}(u, v), \quad \bar{y} = \bar{y}(u, v), \quad u = \bar{u}, \quad v = \bar{v}.$$

If this be a curve, then $\dfrac{dy}{dx}$ must be independent of $\dfrac{dv}{du}$, which involves the condition

$$(5) \qquad \frac{\partial(x, y)}{\partial(u, v)} \equiv 0.$$

Conversely, if this equation holds for all real values of u, v, since only analytic functions are involved, it holds for all values, real or complex. Hence there is a functional relation between x and y, or we have a curve.

THEOREM 1. *A necessary and sufficient condition that the congruence* (4) *should be a curve is that the equation* (5) *should hold for all real values of the parameters u and v.**

If the congruence be given by two equations

$$F(x, y, \bar{x}, \bar{y}) = \bar{F}(\bar{x}, \bar{y}, x, y) = 0,$$

it will represent a curve when, and only when, it is possible to eliminate $d\bar{x}$ and $d\bar{y}$ from the differentials of these expressions, i.e.

$$\frac{\partial(F, \bar{F})}{\partial(x, y)} = \frac{\partial(\bar{F}, F)}{\partial(\bar{x}, \bar{y})} = 0, \tag{6}$$

for all points of the congruence.

Suppose that our congruence is not a curve. A point thereof where equation (5) is not satisfied shall be called a "usual" point. The equation of the tangent at such a point will be

$$\left[(\xi - x)\frac{\partial x}{\partial u} + (\eta - y)\frac{\partial y}{\partial u}\right] du + \left[(\xi - x)\frac{\partial x}{\partial v} + (\eta - y)\frac{\partial y}{\partial v}\right] dv = 0.$$

This system of concurrent lines depends linearly on the real homogeneous parameters du, dv. It will clearly cut an arbitrary transversal in a chain of points, and so may properly be called a "chain" of lines.

THEOREM 2. *If a congruence be not a curve, the tangents at a usual point will generate a chain of lines.*

We must now make a short digression into the theory of transformations of the plane. If (x, y) and (x', y') be corresponding points in a transformation, we may either express the coördinates of the latter in terms of the former, or express both in terms of auxiliary variables u, v. The transformation is directly conformal, if

$$\frac{\partial(x, x')}{\partial(u, v)} = \frac{\partial(y, y')}{\partial(u, v)}; \quad \frac{\partial(x, y')}{\partial(u, v)} = -\frac{\partial(y, x')}{\partial(u, v)}. \tag{A}$$

It is inversely conformal, if

$$\frac{\partial(x, x')}{\partial(u, v)} = -\frac{\partial(y, y')}{\partial(u, v)}; \quad \frac{\partial(x, y')}{\partial(u, v)} = \frac{\partial(y, x')}{\partial(u, v)}, \tag{B}$$

* It is not perfectly clear to whom the credit for this theorem belongs. Study, *Ausgewählte Gegenstände der Geometrie*, Berlin, 1911, p. 43, ascribes it to the second section of Segre's *Nuovo campo di ricerche geometriche*, Atti della Reale Accademia delle Scienze di Torino, vol. 25 (1889–90), but only the necessity of the condition seems to be there proved. However, it is certain that Segre was aware of the sufficiency also. Our Theorem 2 is found on p. 437.

as these reduce to the Cauchy-Riemann equations when $x = u$, $y = v$. It is directly equi-areal, if

$$(C) \qquad \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(x', y')}{\partial(u, v)}.$$

It is inversely equi-areal, if

$$(D) \qquad \frac{\partial(x, y)}{\partial(u, v)} = -\frac{\partial(x', y')}{\partial(u, v)}.$$

These six jacobians are connected by the identical relation

$$(7) \qquad \frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(x', y')}{\partial(u, v)} = \frac{\partial(x, x')}{\partial(u, v)}\frac{\partial(y, y')}{\partial(u, v)} - \frac{\partial(x, y')}{\partial(u, v)}\frac{\partial(y, x')}{\partial(u, v)}.$$

3. Laguerre representation; invariants

In studying a congruence of points, there is a very great advantage in keeping in mind the method of representing a complex point devised by Laguerre.* This consists essentially in representing each complex point by the ordered pair of real points at a zero distance therefrom, the conjugate imaginary point being thus represented by the same real pair in reverse order. The representation works without hitch for all finite points. Analytically we have

$$(8) \qquad \begin{aligned} x &= \frac{1}{2}(X + X') + \frac{i}{2}(Y - Y'), \quad y = \frac{1}{2}(Y + Y') - \frac{i}{2}(X - X'), \\ \bar{x} &= \frac{1}{2}(X + X') - \frac{i}{2}(Y - Y'), \quad \bar{y} = \frac{1}{2}(Y + Y') + \frac{i}{2}(X - X'). \end{aligned}$$

There are certain invariants connected with our congruence which are of fundamental importance; they are

$$J + i\,J' = 2\frac{\partial(x, y)}{\partial(u, v)}; \quad J - i\,J' = 2\frac{\partial(\bar{x}, \bar{y})}{\partial u, v)};$$

$$i\,H = \frac{\partial(x, \bar{x})}{\partial(u, v)} + \frac{\partial(y, \bar{y})}{\partial(u, v)}; \quad K = \frac{\partial(x, \bar{y})}{\partial(u, v)} - \frac{\partial(y, \bar{x})}{\partial(u, v)}.$$

Each of these expressions is unaltered in absolute value by a real congruent transformation of the plane, and is merely multiplied by the jacobian if we change real parameters. If we express these invariants in terms of the Laguerre representation, we have

* *Sur l'emploi des imaginaires dans la géométrie, Collected Works*, vol. 2 (1905), pp. 88, 98.

$$[J + i\,J'] = \left[\frac{\partial(X, Y')}{\partial(u, v)} - \frac{\partial(Y, X')}{\partial(u, v)}\right] + i\left[\frac{\partial(X, X')}{\partial(u, v)} + \frac{\partial(Y, Y')}{\partial(u, v)}\right],$$

$$[J - i\,J'] = \left[\frac{\partial(X, Y')}{\partial(u, v)} - \frac{\partial(Y, X')}{\partial(u, v)}\right] - i\left[\frac{\partial(X, X')}{\partial(u, v)} + \frac{\partial(Y, Y')}{\partial(u, v)}\right],$$

(9)

$$H = \left[\frac{(\partial X', Y')}{\partial(u, v)} - \frac{\partial(X, Y)}{\partial(u, v)}\right],$$

$$K = \left[\frac{\partial(X', Y')}{\partial(u, v)} + \frac{\partial(X, Y)}{\partial(u, v)}\right].$$

We find by using (7) twice

$$\left[\frac{\partial(x, \bar{x})}{\partial(u, v)} - \frac{\partial(y, \bar{y})}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, \bar{y})}{\partial(u, v)} + \frac{\partial(y, \bar{x})}{\partial(u, v)}\right]^2 = -H^2 + K^2 - J^2 - J'^2$$

(10)

$$= -\left\{\left[\frac{\partial(X, X')}{\partial(u, v)} - \frac{\partial(Y, Y')}{\partial(u, v)}\right]^2 + \left[\frac{\partial(X, Y')}{\partial(u, v)} + \frac{\partial(Y, X')}{\partial(u, v)}\right]^2\right\}.$$

4. Conformal and equi-areal relations

We have now finished the dreary task of formula grinding, at least for the moment; let us draw some conclusions. Suppose that we have a curve in the complex plane. That by Theorem 1, and by Formula B, will give an inversely conformal transformation of the Laguerre representatives, unless

$$\frac{\partial(X, Y)}{\partial(u, v)}\,\frac{\partial(X', Y')}{\partial(u, v)} = 0.$$

If the first factor vanish, then either (X, Y) is stationary, or moves along a curve. Making the second assumption we may take u as the single real parameter on which x and y depend; then $\frac{\partial X}{\partial v} = \frac{\partial Y}{\partial v} = 0$, and either $\frac{\partial X'}{\partial v} = \frac{\partial Y'}{\partial v} = 0$, which is inadmissible, or $\left(\frac{\partial(X)}{\partial u}\right)^2 + \left(\frac{\partial Y}{\partial u}\right)^2 = 0$, i. e., $\frac{\partial X}{\partial u} = \frac{\partial Y}{\partial u} = 0$ and (X, Y) is fiixed. When (X, Y) is fixed, the curve traced by (x, y) is a minimal line.

Theorem 3. *If a congruence consist in a curve, not a minimal line, the corresponding relation between the Laguerre representatives is inversely conformal.**

* Study, loc. cit., p. 63. This author studies not only the Laguerre representation but also another closely allied, where a complex curve appears as a special type of directly equi-areal transformations.

We have covered the case of a congruence where the invariant $J + iJ'$ is zero. Suppose next that

$$H = K = 0.$$

We have at once from (9)

THEOREM 4. *If a congruence be inversely conformally related to its conjugate, the Laguerre representatives will trace two curves.*

This theorem might also have been predicted from the fact that we pass from a pair of conjugate imaginary points to their Laguerre representatives by a transformation of period two. Turning next to equation (10) we see that the left hand side can vanish only when each of the individual squares does, and the same happens on the right.

THEOREM 5. *A necessary and sufficient condition that the relation between a congruence, not a curve, and its conjugate should be directly conformal, is that the same should be true of the Laguerre representatives.*

THEOREM 6. *If the relation between a congruence and its conjugate be not inversely conformal yet the equation*

$$H = 0$$

subsist, then the corresponding Laguerre representatives are connected by a directly equi-areal relation.

THEOREM 7. *If the relation between a congruence and its conjugate be not inversely conformal, yet the equation*

$$K = 0$$

subsists, then the corresponding Laguerre representatives are connected by an inversely equi-areal transformation.

The congruences where (x, y) and $(\bar{x}, \bar{y})$ trace curves are a special case of those where points invariantly connected with them do so. We mean by invariantly connected points these whose distances from (x, y) and $(\bar{x}, \bar{y})$ bear given ratios to the distances of these points from one another. It is easy to show that the general expression for the coördinates of such a point is

$$x' = \frac{x - r\bar{x}}{1 - r} - \rho\frac{y - \bar{y}}{1 - r}, \quad y' = \frac{\rho(x - \bar{x})}{1 - r} + \frac{y - r\bar{y}}{1 - r}.$$

We find, with a little patience, that

$$\frac{\partial(x', y')}{\partial(u, v)} = \alpha[J + iJ'] + \beta[J - iJ'] + \gamma H + \delta K,$$

where

$$\alpha = \frac{1+\rho^2}{2}, \quad \beta = \frac{r^2+\rho^2}{2}, \quad \gamma = i\rho(r-1), \quad \delta = -(r+\rho^2),$$

with the identical relation

$$\delta^2 - \gamma^2 = 4\alpha\beta.$$

The equation $\frac{\partial(x', y')}{\partial(u, v)} = 0$ will involve

$$\frac{\partial(\bar{x}', \bar{y}')}{\partial(u, v)} = 0,$$

so that

$$\bar{\alpha}[J - iJ'] + \bar{\beta}[J + iJ'] + \bar{\gamma}H + \bar{\delta}K = 0.$$

The two conditions are equivalent if

$$\alpha : \bar{\beta} = \beta : \bar{\alpha} = \gamma : \bar{\gamma} = \delta : \bar{\delta}.$$

When $\rho \neq 0$, this involves $r = \rho/\bar{\rho}$.

The most interesting case is where (x', y') is in the line from (x, y) to $(\bar{x}, \bar{y})$. Here $\rho = 0$.

$$\begin{aligned} &(J + iJ') + r^2(J - iJ') - 2rK = 0, \\ &(J - iJ') + \bar{r}^2(J + iJ') - 2\bar{r}K = 0, \\ &r(r\bar{r} - 1)(J - iJ') = \bar{r}(r\bar{r} - 1)(J + iJ'). \end{aligned} \tag{11}$$

If

$$r\bar{r} - 1 = 0,$$

the point (x', y') is real; the two equations (11) amount to a single equation. An interesting special case is where

$$J + iJ' = J - iJ', \qquad K = 0.$$

Here the two points on the line from (x, y) to $(\bar{x}, \bar{y})$ which form a square with the Laguerre representatives trace two lines. These points are also used to represent the given complex points. If the equation above connecting r and $\bar{r}$ be not fulfilled, we have

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{r}{\bar{r}} \frac{\partial(\bar{x}, \bar{y})}{\partial(u, v)}.$$

THEOREM 8. *If two conjugate imaginary points trace such congruences that a not real point which divides their segment in a constant ratio traces a curve, then corresponding areas traced in the two congruences bear to one another a fixed ratio.*

5. Congruences which include threads of given type

Let us find out under what circumstances a congruence will include a set of threads lying on minimal lines. As (x, y) or $(\bar{x}, \bar{y})$ traces such a thread, one Laguerre representative must remain fixed, or

$$\frac{\partial(X, Y)}{\partial(u, v)} \frac{\partial(\bar{X}'\bar{Y}')}{\partial(u, v)} = 0, \tag{12}$$
$$K^2 - H^2 = 0.$$

We pass to the more difficult question of finding the necessary and sufficient condition that there should be a normal net in our congruence, i. e., two sets of threads cutting at right angles at each point, either of the whole congruence, or of a certain two-dimensional region thereof. We must have two consistent equations.

$$dx\delta x + dy\delta y = 0,$$
$$d\bar{x}\delta\bar{x} + d\bar{y}\delta\bar{y} = 0.$$

In the Laguerre representation we have

$$dX\delta X' + \delta X dX' + dY\delta Y' + \delta Y dY' = 0,$$
$$dX'\delta Y' - dX\delta Y' + \delta X' dY - \delta X dY' = 0.$$

Let us choose X and Y as parameters u and v;

$$dX\left[\frac{\partial X'}{\partial X}\delta X + \frac{\partial X'}{\partial Y}\delta Y\right] + \delta X\left[\frac{\partial X'}{\partial X}dX + \frac{\partial X'}{\partial Y}dY\right] + dY\left[\frac{\partial Y'}{\partial X}\delta X + \frac{\partial Y'}{\partial Y}\delta Y\right]$$
$$+ \delta Y\left[\frac{\partial Y'}{\partial X}dX + \frac{\partial Y'}{\partial Y}dY\right] = 0,$$

$$\delta Y\left[\frac{\partial X'}{\partial X}dX + \frac{\partial X'}{\partial Y}dY\right] - dX\left[\frac{\partial Y'}{\partial X}\delta X + \frac{\partial Y'}{\partial Y}\delta Y\right] + dY\left[\frac{\partial' X}{\partial X}\delta X + \frac{\partial X'}{\partial Y}\delta Y\right]$$
$$- \delta X\left[\frac{\partial Y'}{\partial X}dX + \frac{\partial Y'}{\partial Y}dY\right] = 0.$$

Eliminating δX and δY

$$\left[\left(\frac{\partial X'}{\partial X}\right)^2 + \left(\frac{\partial Y'}{\partial X}\right)^2 - \left(\frac{\partial(X', Y')}{\partial(X, Y)}\right)^2\right]dX^2 + 2\left[\frac{\partial X'}{\partial X}\frac{\partial X'}{\partial Y} + \frac{\partial Y'}{\partial X}\frac{\partial Y'}{\partial Y}\right]dXdY$$
$$+ \left[\left(\frac{\partial X'}{\partial Y}\right)^2 + \left(\frac{\partial Y'}{\partial Y}\right)^2 - \frac{\partial(X', Y')}{\partial(X, Y)}\right]dY^2 = 0.$$

The condition for real roots is

$$\frac{\partial(X', Y')}{\partial(X, Y)}\left[\left(\frac{\partial X'}{\partial X}\right)^2 + \left(\frac{\partial X'}{\partial Y}\right)^2 + \left(\frac{\partial Y'}{\partial X}\right)^2 + \left(\frac{\partial Y'}{\partial Y}\right)^2 - 2\frac{\partial(X', Y')}{\partial(X, Y)}\right] > 0.$$

The second factor is the sum of the squares of two binomials, and will be equal to zero only when the relation between (X, Y) and (X', Y'), or that between (x, y) and $(\bar{x}, \bar{y})$, is directly conformal. As for the first factor, since

$$\frac{\partial(X', Y')}{\partial(u, v)} = \frac{\partial(X', Y')}{\partial(X, Y)} \frac{\partial(X, Y)}{\partial(u, v)},$$

we may write our condition

$$\frac{\partial(X, Y)}{\partial(u, v)} \frac{\partial(X', Y')}{\partial(u, v)} > 0,$$
$$K^2 - H^2 > 0.$$

The next to last inequality shows that in the Laguerre representation, corresponding infinitesimal triangles have the same sign, that is to say, corresponding differentials rotate in the same direction.

THEOREM 9. *A necessary and sufficient condition that a congruence, not a curve, should contain an orthogonal net is that the relation with the conjugate congruence should not be conformal, and that in the Laguerre representation corresponding rotations about corresponding points go in the same sense.*

THEOREM 10. *A congruence which contains a set of minimal threads, or is conformally related to the conjugate, can not contain one and only one normal net.*

Let us next see under what circumstances a congruence can contain a thread whose tangents are parallel to the corresponding tangents in the conjugate congruence. We have here

$$dx\, d\bar{y} - dy\, d\bar{x} = 0,$$

$$\left(\frac{\partial x}{\partial u}\frac{\partial \bar{y}}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial \bar{x}}{\partial u}\right) du^2 + \left(\frac{\partial x}{\partial u}\frac{\partial \bar{y}}{\partial v} + \frac{\partial x}{\partial v}\frac{\partial \bar{y}}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial \bar{x}}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial \bar{x}}{\partial u}\right) du\, dv + \left[\frac{\partial x}{\partial v}\frac{\partial \bar{y}}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial \bar{x}}{\partial v}\right]^2 dv^2 = 0,$$

$$\left[\left(\frac{\partial(x, \bar{y})}{\partial(u, v)}\right)^2 + \left(\frac{\partial(y, \bar{x})}{\partial(u, v)}\right)\right]^2 - 2\left[\frac{\partial(x, \bar{x})}{\partial(u, v)}\frac{\partial(y, \bar{y})}{\partial(u, v)} + \frac{\partial(x, \bar{y})}{\partial(u, v)}\frac{\partial(y, \bar{x})}{\partial(u, v)}\right] \leqq 0.$$

Hence by (7),

THEOREM 11. *A necessary and sufficient condition that a congruence should contain a system of threads where a point moves parallel to its conjugate is*

$$H^2 - J^2 - J'^2 \leqq 0.$$

The method employed to reach (9) will yield, with much less trouble,

THEOREM 12. *A neccesary and sufficient condition that a congruence should be such that in the Laguerre representation there is a system of curves parallel to their corresponding curves is*

$$J^2 - (K^2 - H^2) \geqq 0.$$

THEOREM 13. *A necessary and sufficient condition that a congruence should be such that in the Laguerre representation a system of curves can be found whose tangents are perpendicular to the corresponding tangents to the corresponding curves is*

$$J'^2 - (K^2 - H^2) \geqq 0.$$

THEOREM 14. *If a congruence, not a curve, have a set of threads along which a point moves parallel to its conjugate, then in the Laguerre representation there is a system of curves parallel to those to which they correspond.*

Let us pass to certain slightly different considerations. Is it possible that the distance between two points of a congruence should be independent of the path? This certainly is the case if the congruence be a curve. On the other hand, when this condition is fulfilled, the distance is a function of the end-points, only, and the squared distance element

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

must be the square of an exact differential, or

$$\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 = 0.$$

THEOREM 15. *The only congruences where the distances of points are independent of the paths are curves.*

6. TANGENT LINES AND CHAINS

We saw in Theorem 2 that the tangents at a usual point of a congruence which is not a curve will generate a chain of lines. It is time to look more closely into this figure. Let the equations of the congruence be expressed in power series:

$$\begin{aligned} x &= x(u, v) + (U - u)\frac{\partial x}{\partial u} + (V - v)\frac{\partial x}{\partial v} + \frac{1}{2}\left[(U - u)^2\frac{\partial^2 x}{\partial u^2}\right. \\ &\quad \left. + 2(U - u)(V - v)\frac{\partial^2 x}{\partial u \partial v} + (V - v)^2\frac{\partial^2 x}{\partial v^2}\right] + \cdots, \\ y &= y(u, v) + (U - u)\frac{\partial y}{\partial u} + (V - v)\frac{\partial y}{\partial v} + \frac{1}{2}\left[(U - u)^2\frac{\partial^2 y}{\partial u^2}\right. \\ &\quad \left. + 2(U - u)(V - v)\frac{\partial^2 y}{\partial u \partial v} + (V - v)^2\frac{\partial^2 y}{\partial v^2}\right] + \cdots. \end{aligned} \tag{13}$$

Let us find the intersection with $y' = y(u, v) + \lambda[x' - x(u, v)]$;

$$0 = \left[\frac{\partial y}{\partial u} - \lambda \frac{\partial x}{\partial u}\right](U - u) + \left[\frac{\partial y}{\partial v} - \lambda \frac{\partial x}{\partial v}\right](V - v)$$
$$+ \frac{1}{2}\left\{\left[\frac{\partial^2 y}{\partial u^2} - \lambda \frac{\partial^2 x}{\partial u^2}\right](U - u)^2 + 2\left[\frac{\partial^2 y}{\partial u \partial v} - \lambda \frac{\partial^2 x}{\partial u \partial v}\right](V - u)(V - v)\right.$$
$$\left. + \left[\frac{\partial^2 y}{\partial v^2} - \lambda \frac{\partial^2 x}{\partial v^2}\right](V - v)^2\right\} + \cdots, \tag{14}$$

$$0 = \left[\frac{\partial \bar{y}}{\partial u} - \bar{\lambda} \frac{\partial \bar{x}}{\partial u}\right](U - u) + \left[\frac{\partial \bar{y}}{\partial v} - \bar{\lambda} \frac{\partial \bar{x}}{\partial v}\right](V - v)$$
$$+ \frac{1}{2}\left\{\left[\frac{\partial^2 \bar{y}}{\partial u^2} - \bar{\lambda} \frac{\partial^2 \bar{x}}{\partial u^2}\right](U - u)^2 + 2\left[\frac{\partial^2 \bar{y}}{\partial u \partial v} - \bar{\lambda} \frac{\partial^2 \bar{x}}{\partial u \partial v}\right] dudv\right.$$
$$\left. + \left[\frac{\partial^2 \bar{y}}{\partial v^2} - \bar{\lambda} \frac{\partial^2 \bar{x}}{\partial v^2}\right](V - v)^2\right\} + \cdots.$$

We have two curves in the (U, V) plane, meeting at (u, v). They will be tangent there if

$$\begin{vmatrix} \frac{\partial \bar{y}}{\partial u} - \bar{\lambda} \frac{\partial \bar{x}}{\partial u} & \frac{\partial \bar{y}}{\partial v} - \bar{\lambda} \frac{\partial \bar{x}}{\partial v} \\ \frac{\partial y}{\partial u} - \lambda \frac{\partial x}{\partial u} & \frac{\partial y}{\partial v} - \lambda \frac{\partial x}{\partial v} \end{vmatrix} = 0,$$

$$\lambda\bar{\lambda} \frac{\partial(x, \bar{x})}{\partial(u, v)} - \lambda \frac{\partial(x, \bar{y})}{\partial(u, v)} - \bar{\lambda} \frac{\partial(y, \bar{x})}{\partial(u, v)} + \frac{\partial(y, \bar{y})}{\partial(u, v)} = 0. \tag{15}$$

This equation is unaltered when we replace λ, $\bar{\lambda}$ by their conjugates $\bar{\lambda}$, λ if, and only if

$$K = 0.$$

It is unaltered when we replace λ and $\bar{\lambda}$ by their diametral values $-1/\bar{\lambda}$, $-1/\lambda$ if and only if

$$H = 0.$$

THEOREM 16. *A necessary and sufficient condition that the tangents at an arbitrary usual point of a congruence, not a curve, should have, in pairs, conjugate imaginary slopes, is that the corresponding transformation in the Laguerre representations should be inversely equi-areal; the slopes will be diametral imaginaries in pairs when the transformation is directly equi-areal.*

Let us see whether we can find any osculating tangents which have three-point contact with the congruence. In the (U, V) plane we must have two curves which not only touch, but have the same curvature.

Let the curves be $F(U, V) = \bar{F}(U, V) = 0$;

$$\frac{\partial(F, \bar{F})}{\partial(U, V)} = 0, \quad U = u, \quad V = v,$$

$$\frac{\dfrac{\partial^2 F}{\partial u^2}\left(\dfrac{\partial F}{\partial v}\right)^2 + \dfrac{\partial^2 F}{\partial v^2}\left(\dfrac{\partial F}{\partial u}\right)^2 - 2\dfrac{\partial^2 F}{\partial u \partial v}\dfrac{\partial F}{\partial u}\dfrac{\partial F}{\partial v}}{\left[\left(\dfrac{\partial F}{\partial u}\right)^2 + \left(\dfrac{\partial F}{\partial v}\right)^2\right]^{3/2}}$$

$$= \frac{\dfrac{\partial^2 \bar{F}}{\partial u^2}\left(\dfrac{\partial \bar{F}}{\partial v}\right)^2 + \dfrac{\partial^2 \bar{F}}{\partial v^2}\left(\dfrac{\partial \bar{F}}{\partial u}\right)^2 - 2\dfrac{\partial^2 \bar{F}}{\partial u \partial v}\dfrac{\partial \bar{F}}{\partial u}\dfrac{\partial \bar{F}}{\partial v}}{\left[\left(\dfrac{\partial \bar{F}}{\partial u}\right)^2 + \left(\dfrac{\partial \bar{F}}{\partial v}\right)^2\right]^{3/2}}.$$

For this purpose it is necessary and sufficient that the following expression should be equal to its conjugate:

$$\frac{\begin{aligned}&\left(\frac{\partial^2 y}{\partial u^2} - \lambda\frac{\partial^2 x}{\partial u^2}\right)\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial^2 y}{\partial v^2} - \lambda\frac{\partial^2 x}{\partial v^2}\right)\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right)^2\\ &\quad - 2\left(\frac{\partial^2 y}{\partial u \partial v} - \lambda\frac{\partial^2 x}{\partial u \partial v}\right)\left(\frac{\partial y}{\partial u} - \lambda\frac{\partial x}{\partial u}\right)\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right)\end{aligned}}{\left[\left(\dfrac{\partial y}{\partial u} - \lambda\dfrac{\partial x}{\partial u}\right)^2 + \left(\dfrac{\partial y}{\partial v} - \lambda\dfrac{\partial x}{\partial v}\right)^2\right]^{3/2}}.$$

The general solution of (15) is

$$\lambda = \frac{\dfrac{\partial y}{\partial u} + X\dfrac{\partial y}{\partial v}}{\dfrac{\partial x}{\partial u} + X\dfrac{\partial x}{\partial v}}, \qquad X = \frac{\dfrac{\partial y}{\partial u} - \lambda\dfrac{\partial x}{\partial u}}{\dfrac{\partial y}{\partial v} - \lambda\dfrac{\partial x}{\partial v}},$$

$$\frac{\partial y}{\partial u} - \lambda\frac{\partial x}{\partial u} = X\left(\frac{\partial y}{\partial v} - \lambda\frac{\partial x}{\partial v}\right) = \frac{X\dfrac{\partial(x,y)}{\partial(u,v)}}{\dfrac{\partial x}{\partial u} + X\dfrac{\partial x}{\partial v}},$$

$$\frac{\left(\dfrac{\partial^2 y}{\partial u^2} - \lambda\dfrac{\partial^2 x}{\partial u^2}\right) - 2X\left(\dfrac{\partial^2 y}{\partial u \partial v} - \lambda\dfrac{\partial^2 x}{\partial u \partial v}\right) + X^2\left(\dfrac{\partial^2 y}{\partial v^2} - \lambda\dfrac{\partial^2 x}{\partial v^2}\right)}{\left(\dfrac{\partial y}{\partial v} - \lambda\dfrac{\partial x}{\partial v}\right)\left(1 + X^2\right)^{3/2}}$$

$$= \frac{\left(\dfrac{\partial^2 \bar{y}}{\partial u^2} - \bar{\lambda}\dfrac{\partial^2 \bar{x}}{\partial u^2}\right) - 2X\left(\dfrac{\partial^2 \bar{y}}{\partial u \partial v} - \bar{\lambda}\dfrac{\partial^2 \bar{x}}{\partial u \partial v}\right) + X^2\left(\dfrac{\partial^2 \bar{y}}{\partial v^2} - \bar{\lambda}\dfrac{\partial^2 \bar{x}}{\partial v^2}\right)}{\left(\dfrac{\partial \bar{y}}{\partial v} - \bar{\lambda}\dfrac{\partial \bar{x}}{\partial v}\right)\left(1 + X^2\right)^{3/2}}.$$

Substituting the values of λ and $\bar{\lambda}$, we get the real equation

$$PX^3 + QX^2 + RX + S = 0.$$

This equation will surely have one real root, perhaps three. Hence, at a usual point of such a congruence there is surely one osculating tangent, and there may be three of them. A thread whose tangents osculate the congruence shall be called "asymptotic."

THEOREM 17. *Every congruence, not a curve, contains a system of asymptotic threads of which at least one passes through every usual point.*

There remain the unusual points. These are characterized by the equations

$$\frac{(\partial x, y)}{\partial(u, v)} = \frac{(\partial \bar{x}, \bar{y})}{\partial(u, v)} = 0.$$

Usually these equations are distinct. If, however, they are identical, there is a whole thread of unusual points.

THEOREM 18. *A necessary and sufficient condition that there should be a thread of unusual points is that in the Laguerre representation there should be a curve of points where the transformation is inversely conformal.*

7. THREE-PARAMETER SYSTEMS

There are two natural analytic methods for representing three-parameter point systems, namely

$$x = x(u, v, w), \quad y = y(u, v, w), \quad \bar{x} = \bar{x}(u, v, w), \quad \bar{y} = \bar{y}(u, v, w), \quad u = \bar{u}, \quad v = \bar{v}, \quad w = \bar{w}; \tag{16}$$

$$F(x, y, \bar{x}, \bar{y}) = 0. \tag{17}$$

The fundamental question to be asked about any such system is whether it does or does not contain completely a curve. A chain of lines gives an example of one sort of system, the totality of points such that the sum of the squares of the absolute values of their distances from the axes has an absolute value unity gives an example of the other. Let us find a necessary and sufficient condition that our system (16) should contain a curve. The curve shall be characterized by the equation

$$\phi(u, v, w) = \text{const.}$$

Then

$$\frac{\partial \phi}{\partial u} : \frac{\partial \phi}{\partial v} : \frac{\partial \phi}{\partial w} = \frac{\partial w}{\partial u} : \frac{\partial w}{\partial v} : -1.$$

The jacobian of x and y with regard to u and v should now vanish, or

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} + \dfrac{\partial x}{\partial w}\dfrac{\partial w}{\partial u} & \dfrac{\partial x}{\partial v} + \dfrac{\partial x}{\partial w}\dfrac{\partial w}{\partial v} \\ \dfrac{\partial y}{\partial u} + \dfrac{\partial y}{\partial w}\dfrac{\partial w}{\partial u} & \dfrac{\partial y}{\partial v} + \dfrac{\partial y}{\partial w}\dfrac{\partial w}{\partial v} \end{vmatrix} = 0,$$

$$\frac{\partial \phi}{\partial u}\frac{\partial(x, y)}{\partial(v, w)} + \frac{\partial \phi}{\partial v}\frac{\partial(x, y)}{\partial(w, u)} + \frac{\partial \phi}{\partial w}\frac{\partial(x, y)}{\partial(w, v)} = 0,$$

$$\frac{\partial \phi}{\partial u}\frac{\partial(\bar{x}, \bar{y})}{\partial(v, w)} + \frac{\partial \phi}{\partial v}\frac{\partial(\bar{x}, \bar{y})}{\partial(w, u)} + \frac{\partial \phi}{\partial w}\frac{\partial(x, y)}{\partial(u, v)} = 0.$$

There will therefore exist a function of u, v, w whose partial derivatives are proportional to the determinants of

$$\left\| \begin{matrix} \dfrac{\partial(x, y)}{\partial(v, w)} & \dfrac{\partial(x, y)}{\partial(w, u)} & \dfrac{\partial(x, y)}{\partial(u, v)} \\ \dfrac{\partial(\bar{x}, \bar{y})}{\partial(v, w)} & \dfrac{\partial(\bar{x}, \bar{y})}{\partial(w, u)} & \dfrac{\partial(\bar{x}, \bar{y})}{\partial(u, v)} \end{matrix} \right\|.$$

Our reasoning is reversible throughout, hence we have:

THEOREM 19. *If x and y be functions of three real parameters, u, v, w, a necessary and sufficient condition that the system so defined should contain one, and, hence, an infinite number of curves is that the Pfaff equation*

$$\begin{vmatrix} du & dv & dw \\ \dfrac{\partial(x, y)}{\partial(v, w)} & \dfrac{\partial(x, y)}{\partial(w, u)} & \dfrac{\partial(x, y)}{\partial(u, v)} \\ \dfrac{\partial(\bar{x}, \bar{y})}{\partial(v, w)} & \dfrac{\partial(\bar{x}, \bar{y})}{\partial(w, u)} & \dfrac{\partial(\bar{x}, \bar{y})}{\partial(u, v)} \end{vmatrix} = 0 \tag{18}$$

should be integrable.

It is time to see what form this condition takes when the system is expressed in the form (17).

Suppose that we have a system of curves, depending on a real parameter R,

$$y = y(x, R), \quad \bar{y} = \bar{y}(\bar{x}, R),$$

which is contained wholly in the variety. We have

$$F(x, y(x, R), \bar{x}, \bar{y}(\bar{x}, R)) \equiv 0.$$

We may treat x and $\bar{x}$ as independent variables and differentiate;

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} = \frac{\partial F}{\partial \bar{x}} + \frac{\partial F}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \bar{x}} = 0,$$

$$\frac{\partial^2 F}{\partial x \partial \bar{x}} + \frac{\partial^2 F}{\partial y \partial \bar{x}}\frac{\partial y}{\partial x} + \frac{\partial^2 F}{\partial x \partial \bar{y}}\frac{\partial \bar{y}}{\partial \bar{x}} + \frac{\partial^2 F}{\partial y \partial \bar{y}}\frac{\partial y}{\partial x}\frac{\partial \bar{y}}{\partial \bar{x}} = 0,$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial x \partial \bar{x}} & \dfrac{\partial^2 F}{\partial x \partial \bar{y}} & \dfrac{\partial F}{\partial x} \\ \dfrac{\partial^2 F}{\partial y \partial \bar{x}} & \dfrac{\partial^2 F}{\partial y \partial \bar{y}} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial \bar{x}} & \dfrac{\partial F}{\partial \bar{y}} & 0 \end{vmatrix} = 0 \quad \text{when } F = 0. \tag{19}$$

Suppose conversely that the first equation holds when the second does. The jacobian in $\bar{x}$ and $\bar{y}$ of F and $\dfrac{\partial F}{\partial x} \Big/ \dfrac{\partial F}{\partial y}$ vanishes, so that F can be written

$$\phi\left(x,\ y,\ \frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}\right) = 0,$$

and this holds when we treat $\bar{x}$ and $\bar{y}$ as variables independent of x and y.

If, then, we give to $\bar{x}$ and $\bar{y}$ such values that

$$\frac{\partial F}{\partial x} = R\frac{\partial F}{\partial y},$$

where R is any real constant, we have ∞^1 curves in our variety, namely

$$\phi(x, y, R) = 0.$$

The invariants of a three-parameter system of points are less simple and interesting than those of a congruence. The squared distance element

$$dx^2 + dy^2$$

is a ternary quadratic form in $du : dv : dw$, but its discriminant is equal to zero, being

$$\left[\frac{\partial(x, y, 0)}{\partial(u, v, w)}\right]^2.$$

Another invariant is the discriminant of

$$dx d\bar{x} + dy d\bar{y}.$$

We find that this is

$$\frac{1}{4}\left[\frac{\partial(x, y, \bar{x})}{\partial(u, v, w)}\frac{(\partial\bar{x}, \bar{y}, x)}{\partial(u, v, w)} + \frac{\partial(x, y, \bar{y})}{\partial(u, v, w)}\frac{\partial(\bar{x}, \bar{y}, y)}{\partial(u, v, w)}\right] \geqq 0.$$

Every three-parameter system of points will have a thread on each minimal line, as we see by counting parameters. The differential equations of these threads are obtained by setting

$$dx \pm i dy = d\bar{x} \pm i d\bar{y} = 0,$$

$$\frac{du}{H(v, w) \pm K(v, w)} = \frac{dv}{H(w, u) \pm K(w, u)} = \frac{dw}{H(u, v) \pm K(u, v)}. \tag{20}$$

8. Tangents

Let us now look at the tangent chains to a three-parameter system. Let us see where chain (3) meets the system (19). Expanding in MacLaurin's series for t we find

$$\begin{aligned} 0 = F(x, y, \bar{x}, \bar{y}) + t\Big\{\rho\Big[(x' - x)\frac{\partial F}{\partial x} + (y' - y)\frac{\partial F}{\partial y}\Big] \\ + \bar{\rho}\Big[(\bar{x}' - \bar{x})\frac{\partial F}{\partial \bar{x}} + (\bar{y}' - \bar{y})\frac{\partial F}{\partial \bar{y}}\Big]\Big\} + \cdots \\ + t^2[\alpha\rho^2 + 2\beta\rho\bar{\rho} + \bar{\alpha}\bar{\rho}^2] + \cdots. \end{aligned} \tag{21}$$

Let us assume that (x, y) is a point of the variety. The equation in t has one root $t = 0$. If, furthermore, (x', y') lies on the straight line

$$(x' - x)\frac{\partial F}{\partial x} + (y' - y)\frac{\partial F}{\partial y} = 0, \tag{22}$$

since the conjugate expression vanishes at the same time, we see that there is another root $t = 0$ regardless of ρ, i. e., every chain from (x, y) to (x', y') is tangent. This line will always exist, unless we are at a singular point where all four partial derivatives vanish. We shall speak of this line as the "tangent" to the variety at the point in question. Note that it will meet the variety in a thread having a double point, or an isolated singularity, at the point of contact. If, on the other hand, (x', y') does not lie on this line, there is just one value of $\rho : \bar{\rho}$ to suit.

Theorem 20. *Every point not on the tangent line to a three-parameter system at a non-singular point may be connected with that point by just one chain touching the variety there.*

THEOREM 21. *The tangent to a three-parameter system at a non-singular point meets the variety in a thread having a double point or isolated singularity there. Every chain of the line through the point of contact is tangent to the variety.*

If we equate both the coefficients of t and of t^2 to zero, and eliminate $\rho : \bar{\rho}$ we have an equation which may be expressed in the symbolic form:

$$[\alpha(x' - x) + \beta(y' - y)]^2 [\bar{\alpha}(x' - x) + \bar{\beta}(\bar{y}' - \bar{y})]^2 = 0.$$

This equation will surely be satisfied by the point (x, y) and if it be satisfied by any other point, then the coördinates of every point of the line from there to (x, y) will satisfy it. We have thus, if we have anything, a one-parameter family of lines through (x, y) and the tangent will be a singular line of the family. If (x', y') be a point on a line of the system, which is not the tangent, the corresponding values of ρ and $\bar{\rho}$ are uniquely determined; when (x', y') is on the tangent, there are two sets of values of $\rho : \bar{\rho}$, usually distinct, which fit.

THEOREM 22. *If there be any point not on the tangent at a non-singular point, whose tangent chain at the non-singular point osculates the three-parameter variety there, then there are ∞^3 such points which generate a one-parameter variety of lines through the given point of the variety, the tangent being singular for the family.*

Let us conclude our work by giving an important geometrical interpretation to equation (19). Let us assume that the origin is the point in which we are interested, and the y axis is the tangent there. We may then express our variety in the form:

$$0 = ay + \bar{a}\bar{y} + (bx^2 + 2cxy + dy^2) + (\bar{b}\bar{x}^2 + 2\bar{c}\bar{x}\bar{y} + \bar{d}\bar{y}^2) + (Bx\bar{x} + \gamma x\bar{y} + \bar{\gamma}\bar{x}y + Dy\bar{y}) + \cdots.$$

Suppose, now, the origin satisfies equation (19) without being a double point of the variety. We easily find

$$B = 0.$$

To find the nature of the thread cut by the variety in its tangent, let us put

$$y = 0, \quad x = X + iY.$$

We have, in the Gauss plane,

$$0 = (B + b\bar{b})X^2 + 2i(b\bar{b})xy + (B - b\bar{b})Y^2 + \cdots.$$

This is a real curve (if it be anything) with the origin as a double point or isolated singularity. The tangents to the two branches will be at right angles if

$$B = 0.$$

If, then, we remember the definition of orthogonal chains given at the beginning of the present article, we reach the curious result:

THEOREM 23. *A necessary and sufficient condition that a three-parameter family of points should contain a one-parameter family of curves is that tangent chains to the two branches of the thread cut by a tangent to the variety should be mutually orthogonal at the point of contact.*

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INVARIANTIVE CHARACTERIZATIONS OF LINEAR ALGEBRAS WITH THE ASSOCIATIVE LAW NOT ASSUMED*

BY

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I. Introduction

We consider linear algebras in which neither the commutative nor the associative law of multiplication is assumed, and whose coördinates and constants of multiplication are numbers of a general field F. A rational integral invariant, or covariant, is a rational integral function of the constants of multiplication, or of the constants of multiplication and the coördinates of the general number, which has the invariantive property under the total group of linear homogeneous transformations on the units. If an invariantive function also actually involves the units, it has been called a vector covariant by Professor O. C. Hazlett,† who shows that every rational integral vector covariant can be obtained as a covariant of the general number of the algebra and a fundamental set of ordinary covariants.

In Section II of this article, it is shown how vector covariants may be formed directly from the constants of multiplication without assuming the knowledge of any ordinary covariants or invariants. To do this, the notion is introduced of a hypercomplex determinant whose elements obey neither the associative nor the commutative law of multiplication, and a few simple properties of such hypercomplex determinants are derived. From the vector covariants and the characteristic determinants of the algebra, ordinary relative invariants may easily be found.

In Section III the linear algebra in three units, one of which is a principle unit, is considered. Invariants and covariants of the algebra are calculated by the method of Section II, and a set of ten of these functions is shown to form a complete system of invariants and covariants from the standpoint of Lie.

In Section IV it is shown that for the example of Section III the arithmetic invariant denoting the rank can be replaced by a rational integral covariant. The generic case is defined as the case for which certain three covariants are

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† These Transactions, vol. 19 (1918), p. 408.

different from zero, and this case is reduced by rational transformations in the general field to a canonical form. The parameters of the canonical form are characterized by invariants.

II. Invariants and vector covariants

1. **Notations.** Consider the general linear algebra in n units whose multiplication table is given by

$$e_i e_j = \sum_{k=1}^{k=n} \gamma_{ijk} e_k \qquad (i, j = 1, \ldots, n), \tag{1}$$

and whose general number is $x = \sum_{i=1}^{i=n} x_i e_i$, where the γ_{ijk} and x_i are numbers of a general field F. Consider the transformation of units

$$T: e_i' = \sum_{j=1}^{j=n} \alpha_{ij} e_j \quad D \equiv |\alpha_{ij}| \neq 0 \qquad (i = 1, \ldots, n), \tag{2}$$

also with coefficients α_{ij} in F. This transformation T induces upon the constants γ_{ijk} of multiplication and the coördinates x_i a transformation S which carries γ_{ijk} into γ_{ijk}' and x_i into x_i' in such a way that

$$e_i' e_j' = \sum_{k=1}^{k=n} \gamma_{ijk}' e_k', \tag{3}$$

and

$$\sum_{j=1}^{j=n} x_j e_j = x = x' = \sum_{i=1}^{i=n} x_i' e_i'. \tag{4}$$

According to Professor Hazlett,* a vector covariant is defined to be a function of the units and the constants of multiplication and the coördinates of the form

$$v(\gamma_{ijk}; x_r; e_s) \qquad (i, j, k, r, s = 1, \ldots, n),$$

such that, under a transformation T of the units and its induced transformation S, there exists a relation of the form

$$v' \equiv v(\gamma_{ijk}'; x_r'; e_s') = D^{\mu} v(\gamma_{ijk}; x_r; e_s) \qquad (i, j, k, r, s = 1, \ldots, n). \tag{5}$$

The integer μ is called the *weight* of the vector covariant v.

As Miss Hazlett noted,† it follows readily from (1) and (3) that every vector covariant is expressible linearly in the units, i. e.,

$$v' = \sum_{i=1}^{i=n} v_i' e_i', \qquad v = \sum_{i=1}^{i=n} v_i e_i, \tag{6}$$

where v_i' is the same function of the γ_{ijk}' and x_r' that v_i is of the γ_{ijk} and the x_r.

* These Transactions, vol. 19 (1918), p. 408.

† Ibid., p. 416.

2. **Theorem 1.** *The coefficients v_i of every vector covariant of weight μ expressed in the form $v = \sum_{i=1}^{i=n} v_i e_i$ are transformed cogrediently apart from the factor $D^{-\mu}$ with the coördinates x_i of the general number $x = \sum_{i=1}^{i=n} x_i e_i$ under a linear transformation of determinant D.*

From (5) and (6) we have

$$\sum_{i=1}^{i=n} v_i' e_i' = D^{\mu} \sum_{j=1}^{j=n} v_j e_j.$$

Making use of (2), we find that

$$\begin{aligned} \sum_{j=1}^{j=n} v_j e_j &= D^{-\mu} \sum_{i=1}^{i=n} v_i' \sum_{j=1}^{j=n} \alpha_{ij} e_j \\ &= \sum_{j=1}^{j=n} D^{-\mu} \sum_{i=1}^{i=n} v_i' \alpha_{ij} e_j. \end{aligned}$$

On account of the linear independence of the units e_j, which occur only where shown explicitly,

$$v_j = D^{-\mu} \sum_{i=1}^{i=n} v_i' \alpha_{ij} \qquad (j = 1, \ldots, n). \tag{7}$$

Similarly from (4) and (2) we have

$$x_j = \sum_{i=1}^{i=n} x_i' \alpha_{ij} \qquad (j = 1, \ldots, n). \tag{8}$$

Comparing (7) with (8), we see that the theorem is proved.

3. **Hypercomplex determinants.*** On account of the fact that multiplication is usually neither commutative nor associative, a determinant whose elements are hypercomplex numbers must be defined more precisely than a determinant whose elements are ordinary numbers. We define the general hypercomplex determinant of the nth order

$$D = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} \tag{9}$$

$$[a_{11} a_{22} \ldots a_{nn}]$$

to be the sum of $n!$ terms of the type

$$(-1)^k [a_{1i_1} a_{2i_2} \ldots a_{ni_n}]$$

in which $i_1, i_2, \ldots, i_n$ is an arrangement of $1, 2, \ldots, n$ derived from the natural order by k interchanges. The first subscripts must occur in their natural order in every term. The brackets indicate that the method of grouping the factors is arbitrary, but the same method is to be used in each term. A particular

* Determinants whose elements are quaternions and hence associative but not commutative were considered by Cayley, *On certain results relating to quaternions*, The Philosophical Magazine, vol. 26 (1845) pp. 141–145; and also by C. J. Joly, second edition of Hamilton's *Elements of Quaternions*, vol. 2, Appendix 1, p. 361.

hypercomplex determinant of the nth order is obtained by replacing the brackets inclosing the leading term by a particular grouping of that term, in which case it is understood that every term of the expansion is to be grouped in that way. Thus there are as many particular hypercomplex determinants of the nth order as there are ways of grouping n ordered factors.

4. **Lemma 1.** *Every hypercomplex determinant merely changes sign upon the interchange of any two columns.*

Consider the determinant (9). Its terms may be arranged in $\frac{1}{2}(n!)$ pairs of the type

$$(10)\qquad \begin{aligned}(-1)^k\,[a_{1i_1}\ \dots\ a_{ri}\ \dots\ a_{si_s}\ \dots\ a_{ni_n}],\\ (-1)^{k+1}\,[a_{1i_1}\ \dots\ a_{ri_s}\ \dots\ a_{si_r}\ \dots\ a_{ni_n}],\end{aligned}$$

where the two terms in each pair differ only in having their i_r and i_s subscripts interchanged. Since each is obtainable from the other by one interchange of subscripts, they are opposite in sign. Let us denote D with its rth and sth columns interchanged by D'. Evidently this interchange leaves the first subscripts in their natural order but interchanges the second subscripts i_r and i_s wherever they occur, and so interchanges the absolute values of the terms in each pair (10). Thus every term in D with its sign changed is equal to a term of D' and vice versa. Then $D' = -D$. This argument depends in no way upon the manner of grouping, so the lemma holds for all hypercomplex determinants.

No analogous theorem exists concerning the interchange of two rows.

Lemma 2. *Any hypercomplex determinant two of whose columns are identical is zero.*

For by the interchange of the two identical columns the determinant is unaltered, and yet changes in sign.

5. **Lemma 3.** *A hypercomplex determinant the elements of whose jth column are binomials $a_{ij} + b_{ij}$ is equal to the sum of two determinants identical with the first except that the jth column of the one is composed of the a_{ij} while the jth column of the other is composed of the b_{ij}. The corresponding theorem holds for rows.*

Let us set

$$D = \begin{vmatrix} a_{11} & \dots & (b_{1r} + c_{1r}) & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & (b_{nr} + c_{nr}) & \dots & a_{nn} \end{vmatrix},$$

$$D_1 = \begin{vmatrix} a_{11} & \dots & b_{1r} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & b_{nr} & \dots & a_{nn} \end{vmatrix}, \qquad D_2 = \begin{vmatrix} a_{11} & \dots & c_{1r} & \dots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & c_{nr} & \dots & a_{nn} \end{vmatrix},$$

where it is understood that the manner of grouping is arbitrary, but the same manner must be used consistently in every term of each determinant. It follows from the distributive law that

$$(11)\quad a_{1i_1} \cdots (b_{si_s} + c_{si_s}) \cdots a_{ni_n} = a_{1i_1} \cdots b_{si_s} \cdots a_{ni_n} + a_{1i_1} \cdots c_{si_s} \cdots a_{ni_n}$$

for all values of the subscripts and for all methods of grouping. It holds in particular for $i_s = r$ and $s = 1, \ldots, n$ where r is fixed. Let k denote the number of interchanges necessary to obtain the order $i_1, \ldots, i_n$ from the natural order $1, \ldots, n$. Then, multiplying (11) by $(-1)^k$ and summing for $i_1, \ldots, i_n$, we have

$$D = D_1 + D_2,$$

which was to be proved.

We prove the corresponding theorem for rows by noting that (11) holds when i_s is fixed and i_s ranges over the values $1, \ldots, n$, and summing as before.

6. **Theorem 2.** *The determinant*

$$V \equiv \begin{vmatrix} e_1 & \cdots & e_n \\ \cdot & \cdot & \cdot \\ e_1 & \cdots & e_n \end{vmatrix}$$

vector covariant of weight 1 *of the linear algebra in n units* $e_1, \ldots, e_n$ *for every manner of grouping.*

We set

$$V' \equiv \begin{vmatrix} e'_1 & \cdots & e'_n \\ \cdot & \cdot & \cdot \\ e'_1 & \cdots & e'_n \end{vmatrix}.$$

Under transformation (2) this becomes

$$V' = \begin{vmatrix} \sum_{i_1=1}^{i_1=n} \alpha_{1i_1} e_{i_1} & \cdots & \sum_{i_n=1}^{i_n=n} \alpha_{ni_n} e_{i_n} \\ \cdot & \cdot & \cdot \\ \sum_{i_1=1}^{i_1=n} \alpha_{1i_1} e_{i_1} & \cdots & \sum_{i_n=1}^{i_n=n} \alpha_{ni_n} e_{i_n} \end{vmatrix}.$$

From Lemma 3 we see that

$$V_1 = \sum_{i_1, \ldots, i_n=1}^{n} \begin{vmatrix} \alpha_{1i_1} e_{i_1} & \cdots & \alpha_{ni_n} e_{i_n} \\ \cdot & \cdot & \cdot \\ \alpha_{1i_1} e_{i_1} & \cdots & \alpha_{ni_n} e_{i_n} \end{vmatrix}.$$

By Lemma 2 such of these determinants as have two identical columns are zero. Moreover the α's are numbers of the field F. Hence we may restrict the $i_1, \ldots, i_n$ to sets of n distinct values and write

$$V' = \sum_{i_1, \ldots, i_n} \alpha_{1i_1} \cdots \alpha_{ni_n} \begin{vmatrix} e_{i_1} & \cdots & e_{i_n} \\ \cdot & \cdot & \cdot \\ e_{i_1} & \cdots & e_{i_n} \end{vmatrix}.$$

Let k denote the number of interchanges necessary to produce the order $i_1, \ldots, i_n$ from the natural order of the integers $1, \ldots, n$. By Lemma 1 the above determinant changes sign with each such interchange, that is with each interchange of two columns. Then

$$V' = \sum_{i_1, \ldots, i_n} (-1)^k \alpha_{1i_1} \cdots \alpha_{ni_n} \begin{vmatrix} e_1 & \cdots & e_n \\ \cdot & \cdot & \cdot \\ e_1 & \cdots & e_n \end{vmatrix}.$$

The above sum is exactly the determinant D of the transformation (2). Hence, as stated,

$$V' = DV.$$

7. **Rational integral invariants.** In the general linear algebra in n units the right- and left-hand characteristic determinants*

$$\begin{aligned} \delta(\omega) &= \left| \textstyle\sum_{i=1}^{i=n} \gamma_{ijk} x_i - \delta_{jk} \omega \right|, \\ \delta'(\omega) &= \left| \textstyle\sum_{j=1}^{j=n} \gamma_{ijk} x_j - \delta_{ik} \omega \right| \qquad (i, j, k = 1, \ldots, n), \end{aligned} \tag{12}$$

and the coefficients of the powers of ω therein are absolute covariants under the general group of homogeneous linear transformations on the units. In view of Theorem 2, there are conceivably as many vector covariants of weight 1 as there are ways of grouping n ordered factors, but it may happen in certain algebras that a smaller number results. Thus for commutative algebras they all vanish and for associative algebras they become identical. By means of the multiplication table (1) each such vector covariant can be expressed linearly in terms of the units, i. e., in the form (6). Suppose that p of them are linearly independent. By Theorem 1 the coefficients v_i are transformed cogrediently apart from the factor D^{-1} with the coördinates x_i of the general number. Therefore if these coefficients v_i be inserted in place of the x_i in a homogeneous absolute invariant of degree r, there results a relative invariant of the linear algebra of weight r. This method yields $2np$ relative invariants.

III. A Lie complete system of invariants and covariants for a ternary algebra

8. **Foreword.** It will be shown that for the algebra with three units, one of which is a principal unit, the invariants formed by the method of the preceding section together with those obtained from the characteristic determinants by

* L. E. Dickson, *Linear Algebras*, Cambridge Tract No. 16 (Cambridge, 1914), p. 17 and p. 20.

the methods of the theory of invariants of forms constitute a complete system of invariants and covariants from the standpoint of Lie. In fact, if all the invariants obtainable by both methods are taken, there is considerable redundance. A set of the simpler ones is shown to be composed of functionally independent invariants.

9. **The finite transformations.** In the linear algebra in three units $1, e_1, e_2$, where 1 is a principal unit, the general number is of the form

$$x = x_0 + x_1 e_1 + x_2 e_2 \tag{13}$$

where x_0, x_1, x_2 are numbers of a field F. The multiplication table may be taken to be

$$\begin{aligned} e_1^2 &= a_0 + a_1 e_2 + a_2 e_2, \\ e_2^2 &= b_0 + b_1 e_1 + b_2 e_2, \\ e_1 e_2 &= c_0 + c_1 e_1 + c_2 e_2, \\ e_2 e_1 &= d_0 + d_1 e_1 + d_2 e_2, \end{aligned} \tag{14}$$

where the coefficients are numbers of F. By applying the linear transformation of units

$$\begin{aligned} e_1' &= \alpha_0 + \alpha_1 e_1 + \alpha_2 e_2, \\ e_2' &= \beta_0 + \beta_1 e_1 + \beta_2 e_2, \end{aligned} \tag{15}$$

where the coefficients are numbers of F such that $D \equiv \alpha_1\beta_2 - \alpha_2\beta_1$ is different from zero, the principal unit is left invariant and the general number and multiplication table become (13) and (14) respectively with each letter primed. That is, the transformation (15) of units induces upon the coefficients the following transformation:

$$\begin{aligned}
x_0' &= x_0 + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)x_1 + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)x_2, \\
x_1' &= D^{-1}\beta_2 x_1 - D^{-1}\beta_1 x_2, \\
x_2' &= -D^{-1}\alpha_2 x_1 + D^{-1}\alpha_1 x_2, \\
a_0' &= A + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)B + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)C, \\
a_1' &= D^{-1}\beta_2 B - D^{-1}\beta_1 C, \\
a_2' &= -D^{-1}\alpha_2 B + D^{-1}\alpha_1 C, \\
b_0' &= E + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)F + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)G, \\
b_1' &= D^{-1}\beta_2 F - D^{-1}\beta_1 G, \\
b_2' &= -D^{-1}\alpha_2 F + D^{-1}\alpha_1 G, \\
c_0' &= H + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)I + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)J, \\
c_1' &= D^{-1}\beta_2 I - D^{-1}\beta_1 J, \\
c_2' &= -D^{-1}\alpha_2 I + D^{-1}\alpha_1 J, \\
d_0' &= K + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)L + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)M, \\
d_1' &= D^{-1}\beta_2 L - D^{-1}\beta_1 M, \\
d_2' &= -D^{-1}\alpha_2 L + D^{-1}\alpha_1 M,
\end{aligned} \tag{16}$$

where

$$
\begin{aligned}
A &\equiv \alpha_0^2 + \alpha_1^2 a_0 + \alpha_2^2 b_0 + \alpha_1\alpha_2 c_0 + \alpha_1\alpha_2 d_0,\\
B &\equiv \alpha_1^2 a_1 + \alpha_2^2 b_1 + 2\alpha_0\alpha_1 + \alpha_1\alpha_2 c_1 + \alpha_1\alpha_2 d_1,\\
C &\equiv \alpha_1^2 a_2 + \alpha_2^2 b_2 + 2\alpha_0\alpha_2 + \alpha_1\alpha_2 c_2 + \alpha_1\alpha_2 d_2,\\
E &\equiv \beta_0^2 + \beta_1^2 a_0 + \beta_2^2 b_0 + \beta_1\beta_2 c_0 + \beta_1\beta_2 d_0,\\
F &\equiv \beta_1^2 a_1 + \beta_2^2 b_1 + 2\beta_0\beta_1 + \beta_1\beta_2 c_1 + \beta_1\beta_2 d_1,\\
G &\equiv \beta_1^2 a_2 + \beta_2^2 b_2 + 2\beta_0\beta_2 + \beta_1\beta_2 c_2 + \beta_1\beta_2 d_2,\\
H &\equiv \alpha_0\beta_0 + \alpha_1\beta_1 a_0 + \alpha_2\beta_2 b_0 + \alpha_1\beta_2 c_0 + \alpha_2\beta_1 d_0,\\
I &\equiv \alpha_1\beta_0 + \alpha_0\beta_1 + \alpha_1\beta_1 a_1 + \alpha_2\beta_2 b_1 + \alpha_1\beta_2 c_1 + \alpha_2\beta_1 d_1,\\
J &\equiv \alpha_2\beta_0 + \alpha_0\beta_2 + \alpha_1\beta_1 a_2 + \alpha_2\beta_2 b_2 + \alpha_1\beta_2 c_2 + \alpha_2\beta_1 d_2,\\
K &\equiv \alpha_0\beta_0 + \alpha_1\beta_1 a_0 + \alpha_2\beta_2 b_0 + \alpha_2\beta_1 c_0 + \alpha_1\beta_2 d_0,\\
L &\equiv \alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_1\beta_1 a_1 + \alpha_2\beta_2 b_1 + \alpha_2\beta_1 c_1 + \alpha_1\beta_2 d_1,\\
M &\equiv \alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_1\beta_1 a_2 + \alpha_2\beta_2 b_2 + \alpha_2\beta_1 c_2 + \alpha_1\beta_2 d_2.
\end{aligned}
$$

10. **The Lie group.** The generators* of the infinitesimal transformation corresponding to the finite transformation (16) are found to consist of six partial differential equations which it is not necessary to give here in detail. The first term of each equation involves only the x_i, and is the only term in which the x_i occur. Since these equations are generators of an infinitesimal group corresponding to the finite group of transformations (16), they form *a fortiori* a complete system of partial differential equations. The six equations in fifteen variables have nine functionally independent solutions, and these solutions form a complete set of absolute invariants and covariants of the linear algebra (14).

Moreover, if the terms involving x_0, x_1, x_2 in these equations be deleted, there results another complete system of partial differential equations, the generators of the group corresponding to the group of transformations in (16) on the constants of multiplication only. This system has six functionally independent solutions, the six absolute invariants of the linear algebra. They are six of the nine solutions of the first complete system. We see then that there are exactly six absolute invariants and three absolute covariants in a complete system of invariants and covariants for the algebra. There are then no more than seven functionally independent relative invariants. A complete system will be exhibited of three absolute covariants and seven relative invariants whose jacobian does not vanish identically.

11. **Invariants of the characteristic determinants.** The right- and left-hand characteristic determinants (12) become

$$
\begin{aligned}
\delta(\omega) &= -\omega^3 + \Lambda\omega^2 - \Gamma\omega + \Delta,\\
\delta'(\omega) &= -\omega^3 + \Lambda'\omega^2 - \Gamma'\omega + \Delta',
\end{aligned}
$$

* Lie-Scheffers, *Vorlesungen über continuierliche Gruppen*, Leipzig, 1893, p. 716, et seq.

where

$$\Lambda = 3x_0 + (a_1 + c_2)x_1 + (d_1 + b_2)x_2,$$

$$\Gamma = 3x_0^2 + (a_1c_2 - a_2c_1 - a_0)x_1^2 + (b_2d_1 - b_1d_2 - b_0)x_2^2 + (2a_1 + 2c_2)x_0x_1 + (2b_2 + 2d_1)x_0x_2 + (a_1b_2 - a_2b_1 + c_2d_1 - c_1d_2 - d_0 - c_0)x_1x_2,$$

$$\Delta = x_0^3 + (a_2c_0 - a_0c_2)x_1^3 + (b_1d_0 - b_0d_1)x_2^3 + (a_1c_2 - a_2c_1 - a_0)x_0x_1^2 + (a_1 + c_2)x_0^2x_1 + (b_2 + d_1)x_0^2x_2 + (b_2d_1 - b_1d_2 - b_0)x_0x_2^2 + (c_0d_2 - c_2d_0 + a_2b_0 - a_0b_2 + a_0c_1 - a_1c_0)x_1^2x_2 + (b_0d_2 - b_2d_0 + a_0b_1 - a_1b_0 + c_1d_0 - c_0d_1)x_1x_2^2 + (c_2d_1 - c_1d_2 + a_1b_2 - a_2b_1 - c_0 - d_0)x_0x_1x_2, \tag{17}$$

and Λ', Γ', Δ' are obtained from Λ, Γ, Δ respectively by interchanging c_i and d_i, for $i = 0, 1, 2$. Each of these six expressions satisfies the equations of §10 and is an absolute covariant of the algebra.

The invariants of the six ternary forms (17) are invariants of the algebra. Λ and Λ' have no invariants. Γ and Γ' have one invariant each, their hessians. The ternary cubic forms Δ and Δ' each have two relative invariants, the S and T of Aronhold.* But only six of the seven invariants have been accounted for by this means, even if all six should prove to be independent. Thus it is evident that the invariants obtainable as the invariants of the coefficients of the characteristic determinants are not sufficient to form a complete system.

12. **Additional invariants by the method of Section II.** By Theorem 2 of §6, the hypercomplex determinant

$$V \equiv \begin{vmatrix} 1 & e_1 & e_2 \\ 1 & e_1 & e_2 \\ 1 & e_1 & e_2 \end{vmatrix} = e_1e_2 - e_2e_1$$

is a vector covariant of weight 1. By the multiplication table (14) this can be expressed as

$$V = (c_0 - d_0) + (c_1 - d_1)e_1 + (c_2 - d_2)e_2. \tag{18}$$

Now by Theorem 1, §2, the coefficients of the units in (18) are transformed cogrediently apart from the factor D^{-1} with the coefficients x_0, x_1, x_2 of the general number, and hence when substituted for these variables in (17) give relative invariants of the algebra.† Thus we have

$$\begin{aligned}
\Lambda_1 &= 3(c_0 - d_0) + (a_1 + c_2)(c_1 - d_1) + (d_1 + b_2)(c_2 - d_2), \\
\Lambda_1' &= 3(c_0 - d_0) + (a_1 + d_2)(c_1 - d_1) + (c_1 + b_2)(c_2 - d_2), \\
\Gamma_2 &= 3(c_0 - d_0)^2 + (a_1c_2 - a_2c_1 - a_0)(c_1 - d_1)^2 + \cdots, \\
\Gamma_2' &= 3(c_0 - d_0)^2 + (a_1d_2 - a_2d_1 - a_0)(c_1 - d_1)^2 + \cdots, \\
\Delta_3 &= (c_0 - d_0)^3 + (a_2c_0 - a_0c_2)(c_1 - d_1)^3 + \cdots, \\
\Delta_3' &= (c_0 - d_0)^3 + (a_2d_0 - a_0d_2)(c_1 - d_1)^3 + \cdots.
\end{aligned} \tag{19}$$

* Salmon, *Higher Plane Curves*, Dublin, 1897, pp. 191–192.

† Throughout this article the subscript on the symbol for an invariant indicates its weight.

Each of these forms is transformed into a function of itself by the differential operators of §10.

Evidently there are functional relations between the twelve relative invariants which these methods yield, since it has been shown that there can be but seven functionally independent invariants. This redundance makes unnecessary the use of the complicated S and T invariants of Δ and Δ'. The S invariant of $\Delta - \Delta'$ is quite simple however, and will be used.

13. **A complete system of invariants.** It will be shown that the following three covariants and seven invariants form a complete system:

$$\Lambda,\ \Lambda',\ \Theta \equiv \Gamma - \Gamma',\ H_2 \equiv \text{hessian of } \Gamma,\ H_2' \equiv \text{hessian of } \Gamma', \tag{20}$$
$$S_4 \equiv S \text{ of } \Delta - \Delta',\ \Lambda_1,\ \Gamma_2,\ \Gamma_2',\ \Delta_3.$$

Since we have only ten relations in fifteen variables, it is sufficient to show that they are independent when five of these variables are put equal to constants. It is found convenient to set $a_0 = a_1 = c_2 = d_2 = 0$ and $a_2 = 1$. In fact this normalization can be made upon Γ, Γ', Δ, Δ' before H_2, H_2', and S_4 are calculated. The ten invariants (20) then reduce to the fairly simple forms:

$$\begin{aligned}
\Lambda &= 3x_0 + (b_2 + d_1)x_2, \\
\Lambda' &= 3x_0 + (b_2 + c_1)x_2, \\
\Theta &= -(c_1 - d_1)x_1^2 - b_2(c_1 - d_1)x_2^2 - 2(c_1 - d_1)x_0x_2, \\
H_2 &= 4c_1(d_1^2 - b_2d_1 + b_2^2) + 12b_0c_1 - 3b_1^2 - 6b_1(c_0 + d_0) \\
&\quad -3(c_0 + d_0)^2, \\
H_2' &= 4d_1(c_1^2 - b_2c_1 + b_2^2) + 12b_0d_1 - 3b_1^2 - 6b_1(c_0 + d_0) \\
&\quad -3(c_0 + d_0)^2, \\
S_4 &= (c_1 - d_1)^2/81\,[3b_2(c_0 - d_0)^2 + 6(c_0 - d_0)(c_1d_0 - c_0d_1) \\
&\quad + 3b_1(c_0 - d_0)(c_1 - d_1) - (c_1 - d_1)^2(b_2^2 + 3b_0)], \\
\Lambda_1 &= 3(c_0 - d_0), \\
\Gamma_2 &= 3(c_0 - d_0)^2 - c_1(c_1 - d_1)^2, \\
\Gamma_2' &= 3(c_0 - d_0)^2 - d_1(c_1 - d_1)^2, \\
\Delta_3 &= (c_0 - d_0)^3 + c_0(c_1 - d_1)^3 - c_1(c_0 - d_0)(c_1 - d_1)^2.
\end{aligned} \tag{21}$$

14. **Independence of the invariants.** To prove the independence of these ten invariants it is sufficient to prove that the jacobian

$$\frac{\partial(\Lambda,\ \Lambda',\ \Theta,\ H_2,\ H_2',\ S_4,\ \Lambda_1,\ \Gamma_2,\ \Gamma_2',\ \Delta_3)}{\partial(x_0,\ x_1,\ x_2,\ b_0,\ b_1,\ b_2,\ c_0,\ c_1,\ d_0,\ d_1)} \tag{22}$$

does not vanish identically. Now only the first three of these polynomials (21) involve x_0, x_1, x_2, and only the first six involve b_0, b_1, b_2. Hence it follows from considering the Laplace development that the jacobian (22) factors into three factors, viz.,

$$\frac{\partial(\Lambda, \Lambda', \Theta)}{\partial(x_0, x_1, x_2)} \times \frac{\partial(H_2, H_2', S_4)}{\partial(b_0, b_1, b_2)} \times \frac{\partial(\Lambda_1, \Gamma_2, \Gamma_2', \Delta_3)}{\partial(c_0, c_1, d_0, d_1)}.$$

It is sufficient to show that no one of these three jacobians vanishes identically in c_1, and hence it is sufficient to show that the coefficient of the highest power of c_1 in each jacobian is not identically zero. It is then permissible to drop all terms in each element except those involving c_1 to the highest power to which it occurs in that element; for the other terms evidently cannot enter into the term of highest degree in c_1 in the expansion of the determinant. It is important to be sure, however, that the coefficient of the highest power does not vanish, for this method does not give the coefficients of lower powers correctly. By this method it is easy to show that

$$\frac{\partial(\Lambda, \Lambda', \Theta)}{\partial(x_0, x_1, x_2)} = 6x_1c_1^2 + \text{lower powers of } c_1,$$

$$\frac{\partial(H_2, H_2', S_4)}{\partial(b_0, b_1, b_2)} = -\frac{16}{9} d_1(c_0 - d_0)\, c_1^5 + \text{lower powers of } c_1,$$

$$\frac{\partial(\Lambda_1, \Gamma_2, \Gamma_2', \Delta_3)}{\partial(c_0, c_1, d_0, d_1)} = -9c_1^7 + \text{lower powers of } c_1.$$

Then the jacobian (22) becomes

$$96\, x_1d_1(c_0 - d_0)c_1^{14} + \text{lower powers of } c_1,$$

and the ten polynomials (20) are functionally independent and form a complete system of invariants and covariants of the linear algebra (14) from the standpoint of Lie.

IV. Characterization by invariants of a canonical form

15. **The rank covariant.** Let us consider the algebra in three units 1, e_1, e_2 whose general number and multiplication table are given by (13) and (14) respectively. The rank* of every such algebra is three or two according as every number does not or does satisfy a quadratic equation. This rank is an arithmetic invariant under every linear transformation of units (15). It will be shown that for this example the arithmetic rank invariant can be replaced by a rational covariant.

* There is an equation $\rho(\omega) = 0$ of lowest degree having X as a right-hand (left-hand) root. The degree of this equation is called the right-hand (left-hand) rank of the algebra. (Cf. Dickson, *Linear Algebras*, p. 23.) It can be shown that there is at least one number satisfying no equation of lower degree.

In every algebra of rank 3 with a principal unit, there is some number x which does not satisfy a quadratic equation. Then the powers $1, x, x^2$ are linearly independent and may be taken as the units $1, e_1, e_2$. That is, the multiplication table of every rank 3 algebra in these units has the form (14) with $a_0 = a_1 = 0$, $a_2 = 1$. Conversely, every algebra of this form is of rank 3. Evidently, then, a necessary and sufficient condition that an algebra (14) be of rank 3 is that it be possible to make the above normalization.

From equations (16) it is seen that the conditions on a transformation (15) which shall make $a_0' = a_1' = 0$ and $a_2' = 1$ are

$$\begin{aligned} &A + D^{-1}(\alpha_2\beta_0 - \alpha_0\beta_2)B + D^{-1}(\alpha_0\beta_1 - \alpha_1\beta_0)C = 0, \\ &D^{-1}\beta_2 B - D^{-1}\beta_1 C = 0, \\ &-D^{-1}\alpha_2 B + D^{-1}\alpha_1 C = 1. \end{aligned}$$

These conditions are readily found to be equivalent to

$$\beta_0 = A, \qquad \beta_1 = B, \qquad \beta_2 = C, \tag{23}$$

where A, B, C, given in (16), are polynomials in the α's and the constants of multiplication. Thus the α's can be chosen arbitrarily and the β's calculated from the above relations, provided only that the determinant of the transformation

$$D = a_2\alpha_1^3 + (-a_1 + c_2 + d_2)\alpha_1^2\alpha_2 + (b_2 - c_1 - d_1)\alpha_1\alpha_2^2 - b_1\alpha_2^3 \tag{24}$$

be different from zero. Now α_1 and α_2 can evidently be chosen so that D is not zero unless every coefficient of D vanishes. Then a necessary and sufficient condition that the algebra be of rank 3 is that not every coefficient a_2 etc. of (24) vanish.

It will now be shown that there exists a covariant Φ whose coefficients are the coefficients of (24). From (17) we form the absolute covariant

$$\Gamma' - \Gamma + \tfrac{1}{3}(\Lambda^2 - \Lambda'^2)$$

every term of which involves $(c_1 - d_1)$ or $(c_2 - d_2)$. It was shown in (18) that $(c_0 - d_0)$, $(c_1 - d_1)$, $(c_2 - d_2)$ are transformed cogrediently apart from the factor D^{-1} with x_0, x_1, x_2, so we may substitute x_1 and x_2 respectively for these expressions and obtain a relative invariant of weight -1,

$$\Phi \equiv a_2x_1^3 + (-a_1 + c_2 + d_2)x_1^2x_2 + (b_2 - c_1 - d_1)x_1x_2^2 - b_1x_2^3,$$

whose coefficients are the coefficients of (24). Hence a necessary and sufficient condition that an algebra (14) be of rank 3 is that Φ does not vanish identically.

16. **Determination of a canonical form.** We shall designate as the generic case that case for which neither Φ, $\Gamma_2 - \Gamma_2'$, nor $\Lambda - \Lambda'$ vanishes identically, and reduce this generic case by rational transformations in the general field F to a canonical form—i. e., to a form in which each remaining coefficient is a relative invariant under the most general transformation which does not destroy the normalization. Since Φ is not identically zero, the multiplication table can at once be reduced as in §15. Then from (23) we see that in order to preserve this normalization the β's must obey the conditions:

$$\begin{aligned} \beta_0 &= \alpha_0^2 + \alpha_2^2 b_0 + \alpha_1\alpha_2(c_0 + d_0), \\ \beta_1 &= \alpha_2^2 b_1 + 2\alpha_0\alpha_1 + \alpha_1\alpha_2(c_1 + d_1), \\ \beta_2 &= \alpha_1^2 + \alpha_2^2 b_2 + 2\alpha_0\alpha_2 + \alpha_1\alpha_2(c_2 + d_2). \end{aligned} \tag{25}$$

If $c_2 - d_2$ is not already zero, it is possible to effect a transformation making $c_2' - d_2'$ equal to zero. From (16) it is seen that under every transformation,

$$c_2' - d_2' = -\alpha_2(c_1 - d_1) + \alpha_1(c_2 - d_2). \tag{26}$$

Therefore if we choose α_1 and α_2 satisfying the conditions

$$\begin{aligned} \alpha_1(c_2 - d_2) &= \alpha_2(c_1 - d_1), \\ D = \alpha_1^3 + (c_2 + d_2)\alpha_1^2\alpha_2 + (b_2 - c_1 - d_1)\alpha_1\alpha_2^2 - b_1\alpha_2^3 &\neq 0, \end{aligned} \tag{27}$$

and the β's according to (25), we have the required transformation. Multiplying the second equation of (27) by $(c_2 - d_2)^3$ and eliminating α_1 by means of the first equation, we have

$$\begin{aligned} D = \alpha_1^3[(c_1 - d_1)^3 &+ (c_2 + d_2)(c_2 - d_2)(c_1 - d_1)^2 \\ &+ (b_2 - b_1 - c_1 - d_1)(c_2 - d_2)^3]. \end{aligned}$$

Hence this normalization is possible if and only if the expression in brackets is different from zero. But this expression is precisely the reduced form of the invariant $\Gamma_2' - \Gamma_2$ which we have assumed not zero.

Since the covariant

$$\Lambda - \Lambda' = (c_2 - d_2)x_1 - (c_1 - d_1)x_2$$

does not vanish identically for this case, we know that $c_1 - d_1$ is not zero. From (26) it is then seen that a necessary and sufficient condition on a transformation that shall leave this normalization undisturbed is that $\alpha_2 = 0$. Then conditions (25) require that $\alpha_1 \neq 0$, $\alpha_2 = 0$, $\beta_0 = \alpha_0^2$, $\beta_1 = 2\alpha_0\alpha_1$, $\beta_2 = \alpha_1^2$, and hence $D = \alpha_1^3$. We see from (16) that by (15)

$$c_2' = 3\alpha_0\alpha_1 + \alpha_1^2 c_2, \tag{28}$$

so let us now apply the transformation

$$\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} -c_2/3 & 1 & 0 \\ c_2^2/9 & -2c_2/3 & 1 \end{pmatrix}$$

which reduces c_2' to zero. Dropping accents, we obtain the canonical form of the generic case for rational transformations, viz.,

$$\begin{aligned} e_1^2 &= e_2, \\ e_2^2 &= b_0 + b_1 e_1 + b_2 e_2, \\ e_1 e_2 &= c_0 + c_1 e_1, \\ e_2 e_1 &= d_0 + d_1 e_1, \qquad c_1 - d_1 \neq 0. \end{aligned} \tag{29}$$

It is evident from (28) that for $c_2 = 0$ and $\alpha_1 \neq 0$, a necessary and sufficient condition that $c_2' = 0$ is that $\alpha_0 = 0$. Then the most general transformation which will leave (29) unaltered in form is of the type

$$\begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_1^2 \end{pmatrix}, \quad D = \alpha_1^3.$$

The effect of this transformation on the coefficients of (29) is to make

$$\begin{aligned} &b_0' = \alpha_1^4 b_0, \quad b_1' = \alpha_1^3 b_1, \quad b_2' = \alpha_1^2 b_2, \quad c_0' = \alpha_1^3 c_0, \quad c_1' = \alpha_1^2 c_1, \\ &d_0' = \alpha_1^3 d_0, \quad d_1' = \alpha_1^2 d_1. \end{aligned} \tag{30}$$

Hence no further reduction by rational transformations is possible, although by a transformation in general irrational any one of these coefficients which is not zero can be reduced to unity. Then the other six are parameters which cannot be altered except by a factor which is a root of unity.

17. **Characterization of parameters.** It is seen from (30) that the most that any transformation can do to (29), preserving its form, is to multiply each coefficient by a power of the determinant. It is natural to expect then that each coefficient in (29) is the reduced form of a relative invariant of the original form (14). We shall expect c_0, d_0 and b_1 to be relative invariants of weight 1; c_1, d_1 and b_2 of weight $\frac{2}{3}$; and b_0 of weight $\frac{4}{3}$. We prove that this is so by actually finding these invariants in terms of known invariants. The invariants which characterize c_0, c_1, d_0, d_1 will be denoted by, respectively, C^0, C', D^0, D'.

Making the normalization indicated in (29), we find that the invariants (19) reduce to the forms

$$\begin{aligned} \Lambda_1 &= \Lambda_1' = 3(c_0 - d_0), \\ \Gamma_2 &= 3(c_0 - d_0)^2 - c_1(c_1 - d_1)^2, \\ \Gamma_2' &= 3(c_0 - d_0)^2 - d_1(c_1 - d_1)^2, \\ \Delta_3 &= \Delta_3' = (c_0 - d_0)^3 + c_0(c_1 - d_1)^3 - c_1(c_0 - d_0)(c_1 - d_1)^2. \end{aligned} \tag{31}$$

We have exactly four relations to determine four unknowns. We find c_0 in terms of the reduced invariants (31) and define C^0 to be the corresponding function of the complete invariants, so that C^0 is an invariant which reduces to c_0 under the normalization (29). Similarly we find invariants which reduce to the other parameters. Thus, since $\Gamma_2' - \Gamma_2 \neq 0$,

$$C^0 = \frac{\Delta_3 - \frac{1}{27}\Lambda_1^3 + \frac{1}{3}\Lambda_1(\frac{1}{3}\Lambda_1^2 - \Gamma_2)}{\Gamma_2' - \Gamma_2},$$

$$D^0 = \frac{\Delta_3 - \frac{1}{27}\Lambda_1^3 + \frac{1}{3}\Lambda_1(\frac{1}{3}\Lambda_1^2 - \Gamma_2')}{\Gamma_2' - \Gamma_2},$$

$$C' = \frac{\frac{1}{3}\Lambda_1^2 - \Gamma_2}{(\Gamma_2' - \Gamma_2)^{\frac{2}{3}}}, \qquad D' = \frac{\frac{1}{3}\Lambda_1^2 - \Gamma_2'}{(\Gamma_2^2 - \Gamma_2)^{\frac{3}{3}}}$$

Evidently the weights are as predicted.

18. **Equations defining the remaining parameters.** Let us now consider the problem of finding invariants which characterize b_0, b_1, b_2. In addition to the invariants (31) we use the hessians H_2 and H_2' of Γ and Γ', and the invariants S_4 and T_6 of $\Delta - \Delta'$. Under the normalization (29) we find

(32) (*a*) H_2, (*b*) H_2', (*c*) S_4 as given by (21),
(*d*) $T_6 = 4l^2mp^3 - 3l^2r^2p^2 + 6p^2lqrs - 12mrpq^3 + 8q^3r^3 - 27p^2q^2s^2$,

where

$$l = c_0 - d_0,\ m = b_0(c_1 - d_1) - b_1(c_0 - d_0),\ p = \tfrac{-1}{3}(c_1 - d_1),$$
$$q = \tfrac{-1}{3}(c_1 - d_1),\ s = \tfrac{1}{3}[b_2(c_0 - d_0) + 2c_1d_0 - 2c_0d_1],$$
$$r = \tfrac{-1}{3}b_2(c_1 - d_1).$$

Making use of the relation (33) in (32*c*), we have (34):

(33) $$H_2 - H_2' = 4(c_1 - d_1)(b_2^2 + 3b_0 - c_1d_1);$$

(34) $$b_1(c_0 - d_0)(c_1 - d_1) + b_2(c_0 - d_0)^2 = \tfrac{1}{3}[81\ S_4/(c_1 - d_1) - 6(c_0 - d_0)(c_1d_0 - c_0d_1) + \tfrac{1}{4}(c_1 - d_1)(H_2 - H_2') + c_1d_1(c_1 - d_1)^2].$$

Multiplying (32*b*) by c_1 and subtracting this product from the product of (32*b*) by d_1, we obtain

(35) $$3b_1^2 + 6b_1(c_0 + d_0) + 4b_2c_1d_1 = (d_1H_2 - c_1H_2')/(c_1 - d_1) + 4c_1d_1(c_1 + d_1) - 3(c_0 + d_0)^2.$$

Directly from (33) comes

$$b_2^2 + 3b_0 = \frac{1}{(4c_1 - 4d_1)}\,[H_2 - H_2' + 4c_1d_1(c_1 - d_1)]. \tag{36}$$

Now c_0, d_0, c_1, d_1 are the values which the known invariants C^0, D^0, C', D' take when the general algebra becomes (29). Consider functions B^0, B', B'' of the coefficients of the general algebra such that B^0 has the value b_0, B' the value b_1 and B'' the value b_2 when the general algebra becomes (29). Then corresponding to (34), (35), (36), and (32d), we have

$$\begin{aligned}
&B'(C^0 - D^0)(C' - D') + B''(C^0 - D^0) = E,\\
&3B'^2 + 6B'(C^0 + D^0) + 4B''C'D' = F,\\
&B''^2 + 3B^0 = G,\\
&T_6 = 4\,l^3mp^3 - 3\,l^2r^2p^2 + 6\,p^2lqrs - 12\,mrpq^3 + 8\,q^3r^3 - 27\,p^2q^2s^2,
\end{aligned} \tag{37}$$

where

$$\begin{aligned}
E &= \tfrac{1}{3}\,[81\,S_4/(C' - D')^2 - 6(C^0 - D^0)(C'D^0 - C^0D')\\
&\quad + \tfrac{1}{4}(C' - D')(H_2 - H_2') + C'D'(C' - D')^2],\\
F &= (D'H_2 - C'H_2')/(C' - D') + 4C'D'(C' + D') - 3(C^0 + D^0)^2,\\
G &= \frac{1}{(4C' - 4D')}\,[H_2 - H_2' + 4C'D'(C' - D')],\\
l &= C^0 - D^0, \qquad m = B^0(C' - D') - B'(C^0 - D^0),\\
p &= -\tfrac{1}{3}\,(C' - D'), \qquad q = -\tfrac{1}{3}\,(C' - D'),\\
r &= -\tfrac{1}{3}\,B''(C' - D'), \qquad s = \tfrac{1}{3}\,[B''(C^0 - D^0) + 2C'D^0 - 2C^0D'].
\end{aligned}$$

Equations (37) determine the invariants B^0, B', B'' uniquely. A practical method for determining them is to calculate the two solutions of the first three equations of (37) and retain only the one which satisfies the fourth relation.

Thus the invariants of the linear algebra (14) which we have found are sufficient to isolate and characterize the generic case.

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CURVES INVARIANT UNDER POINT-TRANSFORMATIONS OF SPECIAL TYPE*

BY

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Introduction. The analytic curves which are invariant under an analytic point-transformation with a fixed point and go through the fixed point have been elegantly treated by Lattès,† who showed that for a transformation of a certain general class there exist just two invariant analytic curves. The facts for the transformations which Lattès' methods can not handle are more complicated. In the particular transformations investigated in this paper, of importance because of their relation to the problem of three bodies‡, it turns out that there may be no *formally* invariant curve, a finite number or a single infinity and that, even when a formally invariant series development exists, it is not, as in Lattès' case, always convergent.

1. **Preliminary discussion.** Given the point transformation:

$$Q: \quad x_1 = X(x, y), \; y_1 = Y(x, y), \tag{1}$$

where at least one point, taken without loss of generality as (0,0), remains fixed and where the functions:

$$X(x, y) = \sum_{i+j=1}^{\infty} a_{ij}x^i y^j, \quad Y(x, y) = \sum_{i+j=1}^{\infty} b_{ij}x^i y^j$$

are single-valued, analytic functions of the complex variables x, y in (0,0). We fix our attention on those transformations for which the roots ρ_1, ρ_2 of the characteristic equation:§

$$\begin{vmatrix} X_x-\rho & X_y \\ Y_x & Y_y-\rho \end{vmatrix}_{(0,0)} = 0$$

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* Presented to the Society, September 5, 1918.

† Lattès, *Sur les équations fonctionelles qui définissent une courbe ou une surface invariante par une transformation*, Annali di Matematica, ser. 3, vol. 13 (1906–1907), pp. 1–137.

‡ Levi-Civita, Comptes Rendus, vol. 131, p. 103; *Sopra alcuni criteri di instabilità*, Annali di Matematica, ser. 3, vol. 5 (1901), pp. 221–307.

§ Picard, *Traité d'Analyse*, vol. 3, pp. 23–46; Poincaré, *Sur les courbes définies par les équations différentielles*, 4e Partie, Journal de Liouville, (1886), pp. 173–195; Lattès, loc. cit.

are both equal to unity* and for which not all the coefficients of the quadratic terms are zero, namely the transformations:

$$(2)\quad T: \qquad \begin{aligned} x_1 &= x + \mu(x, y) = x + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j, \\ y_1 &= y + \nu(x, y) = y + \sum_{i+j=2}^{\infty} b_{ij}x^i y^j, \end{aligned}$$

where the a_{ij}, b_{ij}, $i + j = 2$, are not all zero. We propose to determine conditions for the existence of analytic curves through (0,0) invariant under (2) restricting the discussion to those curves for which (0,0) is a regular point.†

If the curve $y = \psi(x)$:

$$(3)\qquad \psi(x) = \sum_{i=1}^{\infty} l_i x^i, \text{ analytic at } x = 0$$

is invariant under the transformation T, given by (2), that is, if $y_1 = \psi(x_1)$ transformed by T^{-1} yields $y = \psi(x)$, we have

$$(4)\qquad \psi(x) + \nu(x,\psi(x)) = \psi(x + \mu(x,\psi(x))).$$

By equating coefficients of corresponding powers of x on the two sides of this identity we obtain a set of relations in the coefficients l_i of $\psi(x)$ and in the coefficients a_{ij}, b_{ij} of (2). These relations constitute a set of necessary conditions on the l_i for the existence of an analytic curve or curves $y = \psi(x)$ invariant under (2). Moreover the sets of solutions $l_1, l_2, \ldots$ of these relations determine all the developments (3) formally invariant under (2), that is, determine all the formally invariant curves.

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* The general transformations (1), as far as the linear terms are concerned, fall into three least invariant classes with respect to the group of affine transformations S, of §3, leaving the origin fixed. If the roots ρ_1, ρ_2 are distinct, there exists an S such that the transform of Q by S is of the form:

$$(1a)\qquad x_1 = \rho_1 x + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j, \quad y_1 = \rho_2 y + \sum_{i+j=2}^{\infty} b_{ij}x^i y^j, \quad \rho_1\rho_2 \neq 0.$$

If $\rho_1 = \rho_2 = \rho \neq 0$, there exists an S, transforming (1) into

$$(1b)\qquad x_1 = \rho x + y + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j, \quad y_1 = \rho y + \sum_{i+j=2}^{\infty} b_{ij}x^i y^j, \quad \rho \neq 0,$$

except when (1) is already of the form:

$$(1c)\qquad x_1 = \rho x + \sum_{i+j=2}^{\infty} a_{ij}x^i y^j, \quad y_1 = \rho y + \sum_{i+j=2}^{\infty} b_{ij}x^i y^j, \rho \neq 0.$$

The transformations (1a) have been completely investigated by Lattès except for the cases in which $\rho_1 = \rho_2^n$, $\rho_2 = \rho_1^n$, n a positive integer.

† Study: *Ebene analytische Kurven*, 1911, p. 39.

The first four relations obtained from (4), aside from the trivial relation $l_1 = l_1$, are:

$$
\begin{aligned}
&l_1\alpha_2-\beta_2 = 0,\\
&l_1\alpha_3-\beta_3 + l_2(2\alpha_2 + \alpha_2'l_1-\beta_2') = 0,\\
&l_1\alpha_4-\beta_4 + l_3(3\alpha_2 + \alpha_2'l_1-\beta_2') + l_2^2(2\alpha_2' + \tfrac{1}{2}(\alpha_2''l_1-\beta_2''))\\
&\qquad + l_2(\alpha_2^2 + 2\alpha_3 + \alpha_3'l_1-\beta_3') = 0,\\
&l_1\alpha_5-\beta_5 + l_4(4\alpha_2 + \alpha_2'l_1-\beta_2') + l_2l_3(5\alpha_2' + \alpha_2''l_1 - \beta_2'') + l_2^3\alpha_2''\\
&\qquad + l_3(3\alpha_3 + 3\alpha_2^2 + \alpha_3'l_1-\beta_3') + l_2^2(2\alpha_3' + 2\alpha_2\alpha_2' + \tfrac{1}{2}(\alpha_3''l_1-\beta_3''))\\
&\qquad + l_2(2\alpha_4 + 2\alpha_2\alpha_3 + \alpha_4'l_1-\beta_4') = 0,
\end{aligned}
\tag{I}
$$

where the primes denote differentiation with respect to l_1 and

$$\alpha_i = \sum_{r+s=i} a_{rs}l_1^s, \quad \beta_i = \sum_{r+s=i} b_{rs}l_1^s.$$

The compact formula for the nth relation obtained from (4) is derived in §2. The derivation is complicated and unessential for reading what follows. It is sufficient to note that the formula furnishes an immediate proof that the nth relation contains no l with subscript greater than n and that it involves l_n linearly with the coefficient:

$$
\begin{aligned}
\lambda_n(l_1) = n\alpha_2 + \alpha_2'l_1-\beta_2' = {}& (n+2)a_{02}l_1^2 + ((n+1)a_{11}-2b_{02})l_1\\
&+ na_{20}-b_{11},\quad n > 1.
\end{aligned}
\tag{5}
$$

The first equation in (I) is a cubic in l_1 with coefficients from the quadratic terms of (2). If we set $\omega_1 = a_{03}$, $3\omega_2 = a_{11} - b_{02}$, $3\omega_3 = a_{20} - b_{11}$, $\omega_4 = -b_{20}$, it may be written:

$$L(l_1) = \omega_1l_1^3 + 3\omega_2l_1^2 + 3\omega_3l_1 + \omega_4 = 0. \tag{6}$$

When the ω's are not all zero (6) has at most three distinct roots in l_1; there are then at the origin at most three possible slopes for an invariant curve $y = \psi(x)$. The directions determined by these slopes we shall call the *invariant directions*.

If a root $l_1 = l_{11}$ of (6) is substituted in the second relation in (I) and $\lambda_2(l_{11}) \neq 0$, l_2 is uniquely determined: $l_2 = l_{21}$; if this process is continued and always $\lambda_n(l_{11}) \neq 0$, the coefficients l_i in (3) are all uniquely determined.

THEOREM. *If l_{11} is a root of* (6) *and not a root of $\lambda_n(l_1) = 0$, $n > 1$, there is one and only one formally invariant curve tangent at the origin to the invariant direction of slope l_{11}. If, moreover,* (6) *has three distinct roots and no one of them satisfies $\lambda_n(l_1) = 0$, $n > 1$, there are precisely three formally invariant curves through the origin.*

It may happen that $\lambda_i(l_{11}) = 0$; if then the ith relation is not illusory, $l_1 = l_{11}$ leads to a contradiction; if the ith relation is illusory, l_i may be undetermined

and there is a possibility of ∞^1 formally invariant curves all tangent at the origin to the invariant direction of slope l_{11}.

It is conceivable that l_{11}, l_{12}, l_{13} might all lead to contradictions. We shall show in §4, however, that when (6) has three distinct roots there is always one which yields a formally invariant curve. A simple root of (6) can, as we shall prove, satisfy at most one equation $\lambda_n(l_1) = 0$, while a multiple root either satisfies no equation $\lambda_n(l_1) = 0$ and hence yields a single formal ψ-series, or it satisfies every equation $\lambda_n(l_1) = 0$ and various possibilities arise. If the cubic (6) is illusory, $\lambda_n(l_1) = 0$ is satisfied by at most one value of l_1 and this is the same for every n; hence we may say that for "almost every value of l_1" there is a formally invariant curve.

Some of the difficulties involved in proving a formally invariant series $\Sigma_{i=1}^{\infty} l_i x^i$ actually convergent are evident from the discussion in §6 of a particular transformation, for which a formally invariant series always exists but is convergent only when very special conditions are fulfilled.

2. **The general formula for the determination of the coefficients in the power series development of an invariant curve.** In order to write in compact form the general relation (I), we introduce a special notation. Given t_1, t_2, t_3, ..., where we assume that the weight of t_n is $n - 1$ and that the weight of a product in the t's is the sum of the weights of the factors; then by $S_{j,k}(t_i, t_{i+1}, \ldots)$, $i \geqq 1$, we mean the sum of all different ordered products $t_{m_1} t_{m_2} \ldots t_{m_j}$ in t_i, t_{i+1}, ... which are of degree j, j a positive integer, and of weight k, k a positive integer or zero:

$$S_{j,k}(t_i, t_{i+1}, \ldots) = \sum_{m_1 + m_2 + \ldots + m_j = j+k} t_{m_1} t_{m_2} \ldots t_{m_j},$$

where the summation is over all possible different ordered sets: m_1, m_2, ..., m_j, such that $m_n \geqq i$ and $\Sigma_{n=1}^{j} m_n = j + k$. If $i > k/j + 1$, there exist no products $t_{m_1} \ldots t_{m_j}$ as defined:

$$S_{j,k}(t_i, t_{i+1}, \ldots) = 0 \quad (i > k/j + 1);$$

in case $i \leqq k/j + 1$, products $t_{m_1} \ldots t_{m_j}$ as defined do exist and for them $i \leqq m_n \leqq k - ij + i + j$, that is,

$$S_{j,k}(t_i, \ldots) = S_{j,k}(t_i, \ldots, t_p) \quad (i \leqq k/j + 1,\ p = k - ij + i + j).$$

We also agree that

$$S_{0,0}(t_i\ t_{i+1}, \ldots) = 1,\ S_{0,k}(t_i, t_{i+1}, \ldots) = 0,\ k \neq 0 \text{ for all } i.$$

If $y = \Sigma_{i=1}^{\infty} l_i x^i$, then the coefficient of x^{j+n} in y^j, $j \geqq 0$, is $S_{j,n}(l_1, \ldots, l_{n+1})$. Putting $y^j = \Sigma_{n=0}^{\infty} S_{j,n}(l_1, \ldots, l_{n+1}) x^{j+n}$ into $x_1 = x + \Sigma_{i+j=2}^{\infty} a_{ij} x^i y^j$ and denoting the total coefficient of x^k in the result by A_k, we have

$$A_1 = 1,\ A_k = \sum_{i+j=2}^{k} a_{ij} S_{j,k-i-j}(l_1, \ldots, l_{k-i-j+1}) \quad k > 1.$$

The jth power of x_1 : $x_1^j = (\Sigma_{k=1}^{\infty} A_k x^k)^j$ is $\Sigma_{n=0}^{\infty} S_{j,n}(A_1, \ldots, A_{n+1}) x^{j+n}$; accordingly the total coefficient of x^k on the right-hand side of (4) is

$$\sum_{j=1}^{k} l_j S_{j,k-j}(A_1, \ldots, A_{k-j+1}).$$

Since $S_{j,0}(A_1) = A_1^j = 1$, this becomes l_1 for $k = 1$, and becomes

$$l_k + \sum_{j=1}^{k-1} l_j S_{j,k-j}(A_1, \ldots, A_{k-j+1}) \text{ for } k > 1.$$

To get the coefficient of x^k on the left-hand side of (4), we have

$$y_1 = y + \sum_{i+j=2}^{\infty} b_{ij} x^i y^j = \sum_{k=1}^{\infty} (l_k + B_k) x^k, \text{ where}$$

$$B_1 = 0,\ B_k = \sum_{i+j=2}^{k} b_{ij} S_{j,k-i-j}(l_1, \ldots, l_{k-i-j+1}),\ k > 1.$$

The general relation (I) may therefore be written:

$$\sum_{j=1}^{k-1} l_j S_{j,k-j}(A_1, \ldots, A_{k-j+1}) = B_k,\ k > 1. \tag{7a}$$

To aid in the expansion of (7a) we next show that

$$S_{j,n}(t_1, \ldots, t_{n+1}) = j! \sum \frac{t_1^{j-i_1}}{(j-i_1)!} \frac{t_2^{i_1-i_2}}{(i_1-i_2)!} \frac{t_3^{i_2-i_3}}{(i_2-i_3)!} \cdots \frac{t_n^{i_{n-1}-i_n}}{(i_{n-1}-i_n)!} \frac{t_{n+1}^{i_n}}{i_n!}, \tag{IIa}$$

where $j \geqq i_1 \geqq i_2 \geqq \cdots \geqq i_n$ and $\Sigma_{k=1}^{n} i_k = n$. For, from the definition of $S_{j,k}(t_i, \ldots)$

$$S_{j,n}(t_1, \ldots, t_{n+1}) = \sum_{i_1=0}^{j} \frac{j!}{(j-i_1)!\, i_1!} t_1^{j-i_1} S_{i_1,n}(t_2, \ldots, t_{n-i_1+2}), \tag{IIb}$$

$$S_{i_1,n}(t_2, \ldots, t_{n-i_1+2}) = \sum_{i_2=0}^{i_1} \frac{i_1!}{(i_1-i_2)!\, i_2!} t_2^{i_1-i_2} S_{i_2,\, n-i_1+i_2}(t_3, \ldots, t_{n-i_1-i_2+3})$$

and so on; hence

$$S_{j,n}(t_1, \ldots, t_{n+1}) = j! \sum_{i_1=0}^{j} \frac{t_1^{j-i_1}}{(j-i_1)!} \sum_{i_2=0}^{i_1} \frac{t_2^{i_1-i_2}}{(i_1-i_2)!} \cdots \sum_{i_n=0}^{i_{n-1}} \frac{t_n^{i_{n-1}-i_n}}{(i_{n-1}-i_n)!} \sum_{i_{n+1}=0}^{i_n} \frac{t_{n+1}^{i_n-i_{n+1}}}{(i_n-i_{n+1})!} S_{i_{n+1}, q}(t_{n+2}, \ldots)$$

where $q = n - \Sigma_{k=1}^{n} i_k + n i_{n+1}$.

Since $S_{j,n}(t_1, \ldots, t_{n+1})$ has no t with subscript greater than $n+1$, $S(t_{n+2}, \ldots)$ is independent of t and is equal to unity; hence both its degree and weight must be zero. Accordingly $i_{n+1} = 0$, $n - \Sigma_{k=1}^{n} i_k = 0$ and the last sum becomes $\frac{t_{n+1}^{i_n}}{i_n!}$. If we use a single summation sign, understanding thereby this, that the summation is taken over all possible ordered values of $i_1, \ldots, i_n$ for which $j \geqq i_1 \geqq i_2 \geqq \cdots \geqq i_n$ and $\Sigma_{k=1}^{n} i_k = n$, we have the formula (IIa).

By means of (IIa) and the formulas for A_k and B_k we could write out the relations (7a). We facilitate this, however, by rewriting, with the aid of (IIb), the relations (7a) and the formulas for A_k and B_k in forms showing the exact manner of their dependence on l_1. We have, namely, for A_k and B_k, denoting differentiation with respect to l_1 by primes:

$$\text{(II}c\text{)} \qquad \begin{aligned} A_k &= \sum_{i=2}^{k} \sum_{j=0}^{m} \frac{1}{j!} \alpha_i^{[j]} S_{j,k-i}(l_2, \ldots, l_{k-i-j+2}), \\ B_k &= \sum_{i=2}^{k} \sum_{j=0}^{m} \frac{1}{j!} \beta_i^{[j]} S_{j,k-i}(l_2, \ldots, l_{k-i-j+2}). \end{aligned} \qquad (k > 1,\ i \geqq 2),$$

where m is the smaller of the two integers, $i, k-i$ and $\alpha_i = \sum_{r+s=i} a_{rs} l_1^s$, $\beta_i = \sum_{r+s=i} b_{rs} l_1^s$

Instead of (7a) we have:

$$\text{(7}b\text{)} \qquad \sum_{i=1}^{m} \sum_{j=1}^{k-i} l_j \frac{j!}{(j-i)!i!} S_{i,k-j}(A_2, \ldots, A_{k-i-j+2}) - B_k = 0, \quad k > 1,$$

where $m = k/2$, if k is even, and $m = (k-1)/2$, if k is odd. This is our final form for the relations for the determination of the l's. It may be expanded by means of (IIb) and by the formula:

$$S_{j,n}(t_2, \ldots, t_{n-j+2}) = j! \sum \frac{t_2^{j-i_2}}{(j-i_1)!} \frac{t_3^{i_1-i_2}}{(i_1-i_2)!} \cdots \frac{t_{n-j+1}^{i_{n-j-1}-i_{n-j}}}{(i_{n-j-1}-i_{n-j})!} \frac{t_{n-j+2}^{i_{n-j}}}{i_{n-j}!} \tag{IId}$$

where $j \geqq i_1 \geqq i_2 \geqq \cdots \geqq i_{n-j}$ and $\Sigma_{k=1}^{n-j} i_k = n - j$. This formula is the analogue of (IIa) when the i of $S_{j,n}(t_i, \ldots)$ has been increased from one to two.

From (IIc), the weight of A_k or B_k in the l's is $k - 2$; therefore the weight of $S_{j,n}(A_i, \ldots)$ in the l's is $n - j$. The weight in the l's of the ith term in (7b) is $(k - j - i) + (j - 1) = k - i - 1$ and this has its largest value $k - 2$ when $i = 1$. Hence (7b) contains no l with subscript greater than $k - 1$ and is, except for the case $k = 2$, linear in l_{k-1}.

The coefficient in (7b) of l_{k-1}, $k > 2$, which we denote by $\lambda_{k-1}(l_1)$, is $\lambda_{k-1}(l_1) = l_1\alpha_2' - \beta_2' + (k-1)\alpha_2 = (k-1)a_{20} - b_{11} + (ka_{11} - 2b_{02})l_1 + (k+1)a_{02}l_1^2$, $k > 2$. The terms $l_1\alpha_2' - \beta_2'$ come from $l_1 S_{1,k-1}(A_2, \ldots, A_k) \equiv l_1 A_k$ and B_k; the term $(k-1)\alpha_2$ is from $(k-1)l_{k-1}S_{1,1}(A_2) \equiv (k-1)l_{k-1}A_2$. In (7b) the terms of weight zero are $l_1\alpha_k - \beta_k$; they come from $l_1 S_{1,k-1}(A_1, \ldots, A_k) \equiv l_1 A_k$ and B_k.

3. **Invariants and invariant systems of equations for the transformations T.** An analytic curve through the origin invariant under T is transformed by an analytic point transformation U leaving the origin fixed:

$$U: \quad \bar{x} = \sum_{i+j=1}^{\infty} k_{ij} x^i y^j, \quad \bar{y} = \sum_{i+j=1}^{\infty} l_{ij} x^i y^j,$$

into an analytic curve through the origin invariant under T' : $T' = U^{-1}TU$. Hence the study of any transformation T' equivalent to T with respect to the transformations U suffices for T and for all transformations equivalent to T, that is, for the least invariant class of transformations with respect to the group U, to which T belongs. The least invariant classes of transformations T, as far as quadratic terms are concerned, are the same with respect to the group of all analytic transformations U as they are with respect to the group of affine transformations:

$$S: \quad \bar{x} = k_{10}x + k_{01}y, \quad \bar{y} = l_{10}x + l_{01}y, \quad \Delta = k_{10}l_{01} - k_{01}l_{10} \neq 0.$$

The equations giving the values of the coefficients $\bar{a}_{ij}$, $\bar{b}_{ij}$ of the quadratic terms of the transformation T': $T' = S^{-1}TS$ in terms of a_{20}, a_{11}, a_{02}, b_{20}, b_{11}, b_{02} constitute the so-called "group of parameters."* If we introduce instead of

* If the transformation T is to be area-preserving, ω_5, ω_6 must vanish. This geometrical significance of ω_5, ω_6 was suggested by Birkhoff's paper, *Stability in dynamics*, read before the Society, September, 1918.

$a_{20}, \ldots, b_{02}$ the following linear combinations of them:

$$\omega_1 = a_{02}, \quad \omega_2 = \frac{a_{11}-b_{02}}{3}, \quad \omega_3 = \frac{a_{20}-b_{11}}{3},$$

$$\omega_4 = -b_{20}, \quad \omega_5 = \frac{a_{11}+2b_{02}}{3}, \quad \omega_6 = -\frac{b_{11}+2a_{20}}{3}$$

so that

$$a_{20} = \omega_3 - \omega_6, \quad a_{11} = \omega_5 + 2\omega_2, \quad a_{02} = \omega_1,$$
$$b_{20} = -\omega_4, \quad b_{11} = -(2\omega_3 + \omega_6), \quad b_{02} = \omega_5 - \omega_2,$$

the group of parameters becomes:

$$\bar{\omega}_1 = \frac{1}{\Delta^2}\left\{k_{10}^3\omega_1 - 3k_{10}^2k_{01}\omega_2 + 3k_{10}k_{01}^2\omega_3 - k_{01}^3\omega_4\right\},$$

$$\bar{\omega}_2 = \frac{1}{\Delta^2}\left\{k_{01}^2(l_{01}\omega_4 - l_{10}\omega_3) + 2k_{01}k_{10}(l_{10}\omega_2 - l_{01}\omega_3) + k_{10}^2(l_{01}\omega_2 - l_{10}\omega_1)\right\},$$

$$\bar{\omega}_3 = \frac{1}{\Delta^2}\left\{l_{10}^2(k_{10}\omega_1 - k_{01}\omega_2) + 2l_{10}l_{01}(k_{01}\omega_3 - k_{10}\omega_2) + l_{01}^2(k_{10}\omega_3 - k_{01}\omega_4)\right\},$$

$$\bar{\omega}_4 = \frac{1}{\Delta^2}\left\{l_{01}^3\omega_4 - 3l_{01}^2l_{10}\omega_3 + 3l_{01}l_{10}^2\omega_2 - l_{10}^3\omega_1\right\},$$

$$\bar{\omega}_5 = \frac{1}{\Delta}\left\{k_{10}\omega_5 + k_{01}\omega_6\right\},$$

$$\bar{\omega}_6 = \frac{1}{\Delta}\left\{l_{01}\omega_5 + l_{10}\omega_5\right\}.$$

The matrix of the infinitesimal transformations of this group, $G_4(x, y, \omega_1, \ldots, \omega_6)$, is essentially the matrix for obtaining the covariants of the binary cubic: $\omega_1 y^3 + 3\omega_2 xy^2 + 3\omega_3 x^2 y + \omega_4 x^3$, where $y = \omega_6$, $x = \omega_5$.* The relative invariants of G_4 are:

$$f_1 = \omega_1\omega_6^3 + 3\omega_2\omega_5\omega_6^2 + 3\omega_3\omega_5^2\omega_6 + \omega_4\omega_5^3,$$
$$f_2 = (\omega_1\omega_3 - \omega_2^2)\omega_6^2 + (\omega_1\omega_4 - \omega_2\omega_3)\omega_5\omega_6 + (\omega_2\omega_4 - \omega_3^2)\omega_5^2,$$
$$f_3 = 6\omega_1\omega_2\omega_3\omega_4 - 4\omega_1\omega_3^3 - 4\omega_2^3\omega_4 - \omega_1^2\omega_4^2 + 3\omega_2^2\omega_3^2;$$

that is, the binary cubic, its hessian and discriminant. The invariant systems of equations for G_4 are found from the matrix in the usual fashion.

The absolute invariants and invariant systems of equations determine the least invariant classes of transformations T, as far as the quadratic terms are concerned. If we denote by $(a_{20}, \ldots, b_{02})$ the homogeneous coördinates of a

* Lie, *Continuierliche Gruppen*, p. 733.

point in five-dimensional space, these least invariant classes of transformations T are represented by the following least invariant spreads in R_5:

$$m_3: \quad f_1 : f_2 : f_3 = a : b : c, \ a, b, c \text{ not all zero};$$

$$m_2: \quad \begin{cases} \Omega_1 = \omega_2\,\omega_6^2 + 2\omega_3\,\omega_5\,\omega_6 + \omega_4\,\omega_5^2, & \omega_5, \omega_6 \text{ not both zero}, \\ \Omega_2 = \omega_1\,\omega_6^2 + 2\omega_2\,\omega_5\,\omega_6 + \omega_3\,\omega_5^2, & \omega_1, \omega_2, \omega_3, \omega_4 \text{ not all zero}, \\ \omega_2\,\omega_6 + \omega_3\,\omega_5 = c\,\omega_5\,\omega_6, & c \text{ arbitrary}; \end{cases}$$

$$m_2': \quad \omega_5 = \omega_6 = 0, \ f_3 = 0, \ \text{rank of} \left\| \begin{matrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_2 & \omega_3 & \omega_4 \end{matrix} \right\| \text{ two};$$

$$m_1: \quad \omega_1 = \omega_2 = \omega_3 = \omega_4 = 0, \ \omega_5, \omega_6 \text{ not both zero};$$

$$m_1': \quad \omega_5 = \omega_6 = 0, \qquad \text{rank of} \left\| \begin{matrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_2 & \omega_3 & \omega_4 \end{matrix} \right\| \text{ one}.$$

These classes remain mutually exclusive, though not necessarily least invariant when all terms of T are used.

4. **The existence of formally invariant curves.** A sufficient condition that there exist along the invariant direction determined by a root of the cubic (6):

$$L(l_1) = \omega_1 l_1^3 + 3\omega_2 l_1^2 + 3\omega_3 l_1 + \omega_4 = 0, \quad \omega_1, \omega_2, \omega_3, \omega_4 \text{ not all zero}, \tag{6}$$

just one formally invariant curve is that the resultant of $L(l_1)$ and $\lambda_n(l_1)$

$$\lambda_n(l_1) = \frac{n+2}{3} L'(l_1) + (n-1)\,(\omega_5 l_1 - \omega_6) = 0 \quad (n = 2, 3, \ldots), \tag{8}$$

does not vanish for $n > 1$, n an integer. This resultant set equal to zero and freed from a factor ω_1* is:

$$N(n) = (n-1)^3 f_1 + 3(n+2)\,(n-1) f_2 + (n+2)^2 f_3 = 0. \tag{9}$$

THEOREM 1. *A sufficient condition that there is in each of the invariant directions determined by the roots of* (6) *just one formally invariant curve is that* (9) *is satisfied by no integer greater than unity.*

In determining whether a particular root of (6) satisfies an equation (8) it is essential to distinguish between simple and multiple roots of (6).

THEOREM 2. *A simple root of* (6) *can satisfy at most one equation* (8). *A multiple root of* (6) *satisfies no equation* (8) *if f_1, f_2, f_3 are not all zero; it satisfies every equation* (8), *if $f_1 = f_2 = f_3 = 0$.*

* If, for T, $\omega_1 = 0$, we have only to take an equivalent transformation whose representative point does not lie in $\omega_1 = 0$. This is always possible when $L(l_1) \not\equiv 0$.

For, if $\lambda_{n_1}(\bar{l}_1) = 0$ and $\lambda_{n_2}(\bar{l}_1) = 0$, $n_1 \neq n_2$, then $L'(\bar{l}_1) = 0$, $\omega_5 \bar{l}_1 - \omega_6 = 0$, and every equation (8) has $\bar{l}_1$ as a root. If $\omega_5 \bar{l}_1 - \omega_6 = 0$ and $\omega_5 = 0$, then $\omega_6 = 0$ and $\bar{l}_1$ is one of the two roots of $L'(l_1) = 0$; if $\omega_5 \neq 0$, $\bar{l}_1 = \omega_6/\omega_5$ and this satisfies $L'(l_1) = 0$. In either case

$$\Omega_2 = \omega_1\omega_6^2 + 2\omega_2\omega_5\omega_6 + \omega_3\omega_5^2 = 0.$$

LEMMA 1. *Two and hence all equations* (8) *have a common root when and only when* $\Omega_2 = 0$. *The common root always satisfies* $L'(l_1) = 0$ *and if* $\omega_5 \neq 0$, *it is* ω_6/ω_5.

A common root of the equations (8), if it is also a root of (6) is, since it satisfies $L'(l_1) = 0$, a multiple root of (6). In this case (9) is satisfied for all integral n, $n \geqq 2$, and hence $f_1 = f_2 = f_3 = 0$; conversely, if (6) has a multiple root and $f_1 = f_2 = f_3 = 0$, this multiple root satisfies every equation (8).

LEMMA 2. *A root of* (6) *satisfies two and hence every equation* (8) *when and only when it is a multiple root of* (6) *and* $f_1 = f_2 = f_3 = 0$.

A simple root of (6) can, therefore, satisfy at most one equation (8). On the other hand, if a multiple root of (6) satisfies one equation (8), then, since it satisfies $L'(l_1) = 0$, it satisfies every equation (8) and $f_1 = f_2 = f_3 = 0$. Hence, if f_1, f_2, f_3 are not all zero, a multiple root of (6) satisfies no equation (8). This completes the proof of Theorem 2.

A simple root $\bar{l}_1$ of (6) either satisfies no equation (8) and hence yields a single formally invariant curve, or it satisfies just one equation (8): $\lambda_m(\bar{l}_1) = 0$. If $\lambda_m(\bar{l}_1) = 0$, then $\lambda^n(\bar{l}_1) \neq 0$, $n \neq m$, and $l_2, l_3, \ldots, l_{m-1}$ are uniquely determined from the first $m - 1$ relations derived from (4). When their values: $\bar{l}_1, \bar{l}_2, \bar{l}_3, \ldots, \bar{l}_{m-1}$ are substituted in the mth relation, it is either contradictory or illusory; if it is illusory, l_m is undetermined and the subsequent relations give for any chosen l_m, since $\lambda_n(\bar{l}_1) \neq 0$, $n \neq m$, unique values of l_{m+r} $(r = 1, 2, \ldots)$.

THEOREM 3. *A necessary and sufficient condition that a simple root of* (6) *yield a unique formally invariant curve is that it does not satisfy an equation* (8). *If it satisfies an equation* (8), *it yields either no formally invariant curve or a single infinity.*

If $\bar{l}_1$ is a multiple root of (6) and $f_1 = f_2 = f_3 = 0$, there is by Theorem 2 an immediate contradiction unless the second relation of (I): $\alpha_3 l_1 - \beta_3 = 0$ is satisfied by $\bar{l}_1$.

THEOREM 4. *If* f_1, f_2, f_3 *are not all zero, a multiple root of* (6) *always gives a formally invariant curve. If* $f_1 = f_2 = f_3 = 0$, *a necessary condition that a multiple root of* (6) *lead to a formally invariant curve is that this root satisfy* $\alpha_3 l_1 - \beta_3 = 0$.

We now proceed to determine when the conditions which we have set up for the existence of formally invariant curves are fulfilled. If f_1, f_2, f_3 are not all zero, there is always at least one formally invariant curve. This follows from

Theorem 4 if (6) has a multiple root. To show that at least one of three distinct roots of (6) leads to a formally invariant curve, we shall prove a number of lemmas.

LEMMA 3. *If f_1, f_2, f_3 are not all zero, then of the roots of* (9) *considered as an equation in n:*

$$(f_1+3f_2+f_3)n^3+3(-f_1+2f_3)n^2+3(f_1-3f_2+4f_3)n-f_1+6f_2+8f_3=0,$$

at most two can be integers greater than unity.

If f_1, f_2, f_3 are not all zero, (9) considered as an equation in n is not illusory. If $f_1+3f_2+f_3=0$ the lemma is obvious; if $f_1+3f_2+f_3\neq 0$ and two of the roots are integers greater than unity: $r_1 \geqq r_2 > 1$, the remaining root is $(r_1+r_2-2)/(r_1r_2-1)$ and hence is less than unity.

According to this lemma the three roots of (6) cannot each satisfy a different equation (8). That two of them can not satisfy $\lambda_n(l_1)=0$, $n>1$, while a third satisfies $\lambda_n(l_1)=0$, $m>1$, $m\neq n$, we shall show by proving that the condition that $L(l_1)$ have $\lambda_n(l_1)$ as a factor is the same as the condition that (9) have a double root.

The condition that $L(l_1)$ contain $\lambda_n(l_1)$ as a factor is that the rank of the matrix

$$\left\|\begin{matrix}(n+2)\omega_1 & 2(n+2)\omega_2+(n-1)\omega_5 & (n+2)\omega_3-(n-1)\omega_6\\ (n+2)\omega_2-(n-1)\omega_5 & 2(n+2)\omega_3+(n-1)\omega_6 & (n+2)\omega_4\end{matrix}\right\| \tag{10}$$

is less than two, that is, that

$$\begin{aligned} d_1 &= 2(n+2)^2d_{23}+(n+2)(n-1)d_{34}+(n-1)^2\omega_6^2=0,\\ d_2 &= (n+2)^2d_{13}+(n+2)(n-1)d_{24}-(n-1)^2\omega_5\omega_6=0,\\ d_3 &= 2(n+2)^2d_{12}+(n+2)(n-1)d_{14}+(n-1)^2\omega_5^2=0, \end{aligned} \tag{10a}$$

where d_{ij} denotes the determinant made up of the ith and jth columns of the matrix:

$$\left\|\begin{matrix}\omega_1 & \omega_2 & \omega_3-\omega_5\\ \omega_2 & \omega_3 & \omega_4 & \omega_6\end{matrix}\right\|.$$

Then

$$\begin{aligned} &\omega_5^2d_1+2\omega_5\omega_6d_2+\omega_6^2d_3=2(n+2)^2f_2+(n+2)(n-1)f_1=0,\\ &d_{12}d_1-d_{13}d_2+d_{23}d_3=(n-1)^2f_2+(n+2)^2f_3=0,\\ &d_{14}d_1-2d_{24}d_2+d_{34}d_3=2(n+2)(n-1)f_2+(n-1)^2f_1=0. \end{aligned} \tag{10b}$$

Hence

$$f_1:f_2:f_3=2(n+2)^3:-(n-1)(n+2)^2:(n-1)^3,\ n>1. \tag{11}$$

Conversely, if $f_1:f_2:f_3$ are given by (11), it follows by retracing steps that the rank of the matrix (10) is less than two.

LEMMA 4. *A necessary and sufficient condition that two of the roots of* (6) *are the roots of* $\lambda_n(l_1) = 0$, $n > 1$, *is that*

$$f_1 : f_2 : f_3 = 2(n+2)^3 : -(n-1)(n+2)^2 : (n-1)^3, \quad n > 1.$$

On the other hand, (9) has a double root, that is, $N(n) = 0$ and $N'(n) = 0$ have a common root, when and only when $f_1 : f_2 : f_3$ are given by (11).

LEMMA 5. *A necessary and sufficient condition that two of the roots of* (6) *are the roots of* $\lambda_m(l_1) = 0$, $m > 1$, *is that* $n = m$ *is a double root of* (9).

This completes the desired proof.

THEOREM 5. *If* f_1, f_2, f_3 *are not all zero there exists one curve through the origin formally invariant under* T.

If $f_1 = f_2 = f_3 = 0$, (6) always has a multiple root, which by Theorem 2 satisfies every equation (8) and, unless conditions depending on the coefficients of terms in T of higher degree than the second are fulfilled, the multiple root yields no formally invariant curve. If $f_2 \neq 0$, that is, if the rank of

$$\left\| \begin{matrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_2 & \omega_3 & \omega_4 \end{matrix} \right\| \tag{12}$$

is two, (6) has also a simple root and this root yields in general a formally invariant curve. For, if the rank of (12) is two and $\omega_5 = \omega_6 = 0$, the simple root $\bar{l}_1$ of (6) satisfies no equation (8). If $\omega_5 \neq 0$, the double root is ω_6/ω_5, and the simple root is

$$\bar{l}_1 = -\frac{3\omega_2\omega_5 + 2\omega_1\omega_6}{\omega_1\omega_5}.$$

If $\lambda_n(\bar{l}_1) = 0$, $n > 1$, we have, using the notation of (10a)

$$n = \frac{\omega_5^2 + 2d_{14}}{\omega_5^2 - d_{14}}, \quad n > 1.$$

Similarly in case $\omega_6 \neq 0$,

$$n = \frac{\omega_6^2 + 2d_{34}}{\omega_6^2 - d_{24}}, \quad n > 1.$$

In place of either one of these ratios we may use the equal ratio:

$$M = \frac{\omega_5^2 + 2(d_{14} + d_{34}) + \omega_6^2}{\omega_5^2 - (d_{14} + d_{34}) + \omega_6^2},$$

which can be shown to be an invariant when $f_1 = f_2 = f_3 = 0$ of the transformation T under the affine transformations S.

THEOREM 6. *If $f_1 = f_2 = f_3 = 0$ and $f_2 \neq 0$, the simple root of (6) yields a formally invariant curve unless M is defined and has a value equal to an integer greater than unity.*

If (6) is illusory, $\lambda_n(l_1) = (n-1)(\omega_5 l_1 - \omega_6)$ and for every value of l_1 except perhaps for the value ω_6/ω_5 there exists a formally invariant curve.

5. **Classification of the transformations T according to the common factors of the quadratic terms.** The least invariant classes of transformations T as far as their quadratic terms:

$$\alpha(x,y) = a_{20}x^2 + a_{11}xy + a_{02}y^2, \quad \beta(x,y) = b_{20}x^2 + b_{11}xy + b_{02}y^2,$$

$\alpha(x, y)$, $\beta(x, y)$ not both identically zero, are concerned, may be classified, on the basis fundamental in Levi-Civita's discussion of stability* into the three types:

TYPE 1. *$\alpha(x,y)$, $(\beta x,y)$ have no common factor in x,y;*
TYPE 2. *$\alpha(x,y)$, $\beta(x,y)$ have one and only one common factor, linear in x,y;*
TYPE 3. *$\alpha(x,y)$, $\beta(x,y)$ are linearly dependent.*

In making the classification we assume $\omega_4 = 0$.† The least invariant spreads in this case are the intersections of the least invariant spreads m_3, m_2, m_2', m_1, m_1', of G_4 with $\omega_4 = 0$:

$$\begin{aligned}
M_2&: && f_1:f_2:f_3 = a:b:c,\ a, b, c \text{ not all zero};\\
M_1&: && \begin{cases} \omega_3 = \omega_6 = 0,\ \omega_2:\omega_5 = g:h,\ g \neq 0;\\ D_1 = D_2 = D_3 = 0,\ \omega_3:\omega_6 = g:h,\ g \neq 0;\end{cases}\\
M_1'&: && \omega_2 = \omega_3 = \omega_6 = 0,\ \omega_1\omega_5 \neq 0;\\
M_1''&: && \omega_1 = \omega_2 = \omega_3 = 0,\ \omega_5, \omega_6 \text{ not both zero};\\
M_0&: && \omega_2 = \omega_3 = \omega_5 = \omega_6 = 0,\ \omega_1 \neq 0;
\end{aligned}$$

where D_i denotes the determinant obtained from the matrix:

$$\left\| \begin{matrix} 2\omega_1 & \omega_2 & -\omega_5 \\ 3\omega_2 & 2\omega_3 & \omega_6 \end{matrix} \right\|$$

by omitting the ith column. In particular, m_3, m_1, m_1' yield, respectively, M_2, M_1'', M_0; the intersection of m_2 with $\omega_4 = 0$ consists of M_1 ($g \neq 0$) and M_1', while that of m_2' is $M_1(g = 0)$. Each of the ∞^1 spreads M_1 is made up of two distinct continua as indicated; by the G_4 a point P on one can be transformed into any

* Levi-Civita, Annali di Matematica, ser. 3, vol. 5 (1901), p. 242. He shows that the transformations:

$$x_1 = x + x^2 + y(ax + by) + U(x,y),\ y_1 = y(1 + cx + dy) + V(x,y)$$

are unstable, except perhaps for $c = 1$.

† If for a given T, $\omega_4 \neq 0$, we have only to take an equivalent transformation for which $\omega_4 = 0$.

point P' on the other, but by the sub-group G_3 with $l_{10} = 0$, for which $\omega_4 = 0$ is an invariant equation, a point P on one cannot be transformed into a point P' on the other.

The least invariant spreads for $\omega_4 = 0$ which are of Type 1 are those for which the resultant: $f_1 + 3f_2 + f_3$ of $\alpha(x,y)$ and $\beta(x,y)$ does not vanish. Those of Type 3 are determined by considering in detail the three cases: $\alpha(x,y) \not\equiv 0$, $\beta(x,y) \not\equiv 0$; $\alpha(x,y) \not\equiv 0$, $\beta(x,y) \equiv 0$; $\alpha(x,y) \equiv 0$, $\beta(x,y) \not\equiv 0$. Those for Type 2 are then obtained by exclusion. The resulting classification of the least invariant spreads for $\omega_4 = 0$, together with a representative transformation for each least invariant spread, is:

Type	Spread	Representative Transformations
(1a)	$M_2(a + 3b + c \neq 0)$	$\begin{cases} x_1 = x + a_{20}x^2 + a_{11}xy + y^2 + \cdots, a_{20} \neq 0 \\ y_1 = y \quad + xy \quad + \cdots \end{cases}$
(2a)	$M_2(a + 3b + c = 0,$ $a:b:c \neq 2:-1:1)$	$\begin{cases} x_1 = x + a_{11}xy + y^2 + \cdots \\ y_1 = y + xy \quad + \cdots \end{cases}$
(2b)	$M_1(g:h \neq -1:2,$ $g:h \neq 1:1)$	$\begin{cases} x_1 = x + a_{11}xy + \cdots, a_{11} \neq 0,1, \\ y_1 = y \quad + y^2 + \cdots \end{cases}$
(2c)	M_1''	$\begin{cases} x_1 = x + xy + \cdots \\ y_1 = y + y^2 + \cdots \end{cases}$
(2d)	M_1'	$\begin{cases} x_1 = x + xy + y^2 + \cdots \\ y_1 = y \quad + y^2 + \cdots \end{cases}$
(3a)	$M_2(a:b:c = 2:-1:1)$	$\begin{cases} x_1 = x + xy + \cdots \\ y_1 = y + xy + \cdots \end{cases}$
(3b)	$M_1(g:h = -1:2)$	$\begin{cases} x_1 = x + \cdots \\ y_1 = y + y^2 + \cdots \end{cases}$
(3c)	$M_1(g:h = 1:1)$	$\begin{cases} x_1 = x + xy + \cdots \\ y_1 = y + \cdots \end{cases}$
(3d)	M_0	$\begin{cases} x_1 = x + y^2 + \cdots \\ y_1 = y + \cdots \end{cases}$

The representative transformations as given for the non-singular least invariant spreads, that is, those for which f_1, f_2, f_3 are not all zero, are easily obtained by using the fact that when $\omega_4 = 0$ these non-singular least invariant spreads are given by:

$$G = a\omega_3^3 + 3b\omega_3^2\omega_6 + c\omega_6^3 = 0,$$
$$H = a\omega_3^2(\omega_2^2 - \omega_1\omega_3) + b\omega_1\omega_3^2\omega_6 - c\omega_3\omega_6(\omega_2\omega_6 + \omega_3\omega_5) = 0,$$

and by showing that it is always possible to take not only $b_{20} = 0$ but also $b_{02} = 0$, $b_{11} = 1$. Those given for the singular least invariant spreads can be picked out at once.

Each of these representative transformations has been chosen without regard to the particular invariant direction brought into coincidence with the axis of x. Hence for each representative transformation we have to consider the possibility of invariant curves not only in the direction of the x-axis but also in the other invariant directions. It is, however, of advantage to have the invariant direction to be discussed always along the x-axis, that is, to supplement each representative transformation by one or more transformations which have, respectively, along the x-axis the non-equivalent types of invariant directions for the least invariant spread in question. If the transformation T as given has an invariant direction l_1, besides that along the x-axis, we transform T into $\overline{T}$, so that $\overline{T}$ has its corresponding invariant direction $\bar{l}_1$, along the x-axis. If each $\overline{T}$ can be taken in the same form as T, then T suffices; otherwise to T must be added transformations representative of the $\overline{T}$ which can not be taken of the form T.

Denoting by the subscripts 1 and 2 two equivalent transformations with non-equivalent invariant directions along the x-axis, we have as our complete classification:

$$(1a)\quad \begin{aligned} x_1 &= x + a_{20}x^2 + a_{11}xy + y^2 + \cdots, \quad a_{20} \neq 0,\ a_{20}a_{11}^2 \neq (2a_{20}-1)^2,\\ y_1 &= y \qquad\quad + xy \qquad\quad + \cdots \end{aligned}$$

$$(1a_1)\quad \begin{aligned} x_1 &= x + a_{20}x^2 + a_{11}xy + y^2 + \cdots, \quad a_{20} \neq 0,\ a_{20}a_{11}^2 = (2a_{20}-1)^2,\\ y_1 &= y \qquad\quad + xy \qquad\quad + \cdots, \end{aligned}$$

$$(1a_2)\quad \begin{aligned} x_1 &= x + x^2 + a_{02}y^2 + \cdots\\ y_1 &= y \qquad\quad + y^2 + \cdots \end{aligned}$$

$$(2a_1)\quad \begin{aligned} x_1 &= x + a_{11}xy + y^2 + \cdots\\ y_1 &= y + xy \qquad\quad + \cdots \end{aligned} \qquad (2a_2)\quad \begin{aligned} x_1 &= x + a_{20}x^2 + xy + \cdots, \quad a_{20} \neq 0,\\ y_1 &= y \qquad\quad + xy + \cdots \end{aligned}$$

$$(2b_1)\quad \begin{aligned} x_1 &= x + a_{11}xy + \cdots, \quad a_{11} \neq 0, 1,\\ y_1 &= y + y^2 + \cdots \end{aligned} \qquad (2b_2)\quad \begin{aligned} x_1 &= x + x^2 + \cdots\\ y_1 &= y + b_{11}xy + \cdots, \quad b_{11} \neq 0,1 \end{aligned}$$

$$(2c_1)\quad \begin{aligned} x_1 &= x + xy + \cdots\\ y_1 &= y + y^2 + \cdots \end{aligned} \qquad (2c_2)\quad \begin{aligned} x_1 &= x + x^2 + \cdots\\ y_1 &= y + xy + \cdots \end{aligned}$$

$$(2d)\quad \begin{aligned} x_1 &= x + xy + y^2 + \cdots\\ y_1 &= y \qquad + y^2 + \cdots \end{aligned}$$

$$(3a_1)\quad \begin{aligned} x_1 &= x + xy + \cdots\\ y_1 &= y + xy + \cdots \end{aligned} \qquad (3a_2)\quad \begin{aligned} x_1 &= x + x^2 + xy + \cdots\\ y_1 &= y + \cdots \end{aligned}$$

$$(3b_1)\quad \begin{aligned} x_1 &= x + \cdots\\ y_1 &= y + y^2 + \cdots \end{aligned} \qquad (3b_2)\quad \begin{aligned} x_1 &= x + x^2 + \cdots\\ y_1 &= y + \cdots \end{aligned}$$

$$(3c_1)\quad \begin{aligned} x_1 &= x + xy + \cdots\\ y_1 &= y + \cdots \end{aligned} \qquad (3c_2)\quad \begin{aligned} x_1 &= x + \cdots\\ y_1 &= y + xy + \cdots \end{aligned}$$

$$(3d)\quad \begin{aligned} x_1 &= x + y^2 + \cdots\\ y_1 &= y + \cdots \end{aligned}$$

These type forms are mutually exclusive; moreover, two transformations of any type with different values for the constants of that type are mutually exclusive.

From a consideration of equation (9) it follows that:

THEOREM: *If the roots of* (6) *are distinct there are at least two curves formally invariant under Type* $1a_i$; *if* $a_{02} = 0$ *there are three. If the roots of* (6) *are distinct there are at least two curves formally invariant under Type* $2a_i$; *if they are not distinct each root yields a formally invariant curve. There are under Type* $3a_i$ *three formally invariant curves.*

In Types $2b_2$, $2c_2$, $3b_2$, $3c_2$ the simple invariant direction is along the x-axis and it is easy to discover, by computing the value of M, when the simple root of (6) gives trouble.

THEOREM. *In Type* $2b_2$, *if* b_{11} *is not an integer greater than two, the simple root of* (6) *yields a formally invariant curve. In Type* $2c_2$, *every root of* (6) *yields a formally invariant curve; hence there are under Type* $2c_2$ ∞^1 *formally invariant curves. In Types* $3b_2$, $3c_2$ *the simple root of* (6) *yields a formally invariant curve.*

6. **Curves invariant under some special transformations.** If the curve $y = \psi(x)$ is invariant under:

$$T: \qquad x_1 = x + f(x) = \frac{x}{1-x}, \qquad y_1 = y + \varphi(x), \quad b_2 = 0, \tag{13}$$

then $\psi(x)$ is a solution, analytic at the origin, of the functional equation:

$$\psi(x) + \varphi(x) = \psi\left(\frac{x}{1-x}\right).$$

The equations (I) here become $l_1 = 0$ and

$$b_{k+1} = \frac{k!}{(k-1)!} l_2 + \cdots + \frac{k!}{2!(k-2)!} l_{k-1} + \frac{k!}{(k-1)!} l_k \quad (k = 2, 3, \ldots).$$

The equations $k = 2, \ldots, k$ solved for l_k give:

$$\frac{1}{(k-1)!} l_k = \frac{1}{k!} b_{k+1} + M_1 \frac{1}{(k-1)!} b_k + \cdots + M_{k-2} \frac{1}{2!} b_3 \quad (k = 2, 3, \ldots),$$

where the M's satisfy the relations:

$$\frac{1}{(k-1)!} + \frac{M_1}{(k-2)!} + \frac{M_2}{(k-3)!} + \cdots + M_{k-2} = 0 \quad (k = 3, 4, \ldots),$$

that is, are the coefficients in the power series development of the function $\frac{x}{e^x-1}$. Hence

$$\sum_{n=2}^{\infty}\frac{1}{(k-1)!}l_k x^{k-1}=\sum_{k=2}^{\infty}\left(\frac{1}{k!}b_{k+1}+M_1\frac{1}{(k-1)!}b_k+\cdots+M_{k-2}\frac{1}{2!}b_3\right)x^k$$

$$=\frac{x}{e^x-1}\ \frac{1}{x}\sum_{k=2}^{\infty}\frac{b_k x^{k-1}}{(k-1)!}.$$

If for an arbitrary series $\eta(x)=\Sigma_{n=2}^{\infty}v_n x^n$, we define the series $\eta_1(x)=\sum_{n=2}^{\infty}\frac{v_n x^{n-1}}{(n-1)!}$ as the *associated* η-*series*, this relation becomes:

$$\varphi_1(x)=(e^x-1)\psi_1(x)^*$$

or since $f_1(x)=\sum_{n=2}^{\infty}\frac{x^{n-1}}{(n-1)!}=e^x-1$

$$\varphi_1(x)=f_1(x)\psi_1(x). \tag{14}$$

In order to determine when, for $f(x)=\Sigma_{n=2}^{\infty}x^n$ and a given $\varphi(x)$ convergent for $|x|<\rho$, there exists a convergent $\psi(x)$ satisfying (14) we shall have to discuss in detail the relationship of a series $\eta(x)$ to its associated series with respect to convergence and divergence. Since the series $\Sigma_{n=2}^{\infty}v_n x^{n-1}$ converges and diverges for the same x as $\eta(x)$, we may equally well consider its relationship to $\eta_1(x)$ or more simply, the *relationship of* $\vartheta(x)=\Sigma_{n=1}^{\infty}n!\,u_n x^n$ and $\theta_1(x)=\Sigma_{n=1}^{\infty}u_n x^n$.

If $\vartheta(x)$ converges, $\theta_1(x)$ converges for all x and is an integral function; if $\theta_1(x)$ is an integral function, we make use of the classification of integral functions according to their strength of convergence, or, more precisely according to their λ-, μ-, ν-*indices*† in order to determine the behavior of $\theta(x)$.

The λ-index of the integral function $\theta_1(x)$, whose zeros are: $c_1, c_2, \ldots, |c_1|\leqq|c_2|\leqq\ldots$, where $x=0$, if present, has been excluded, is, if it exists, the number $\lambda\geqq 0$, such that, for every positive ϵ, $\sum_{n=1}^{\infty}\frac{1}{|c_n|^{\lambda+\epsilon}}$ converges, whereas $\sum_{n=1}^{\infty}\frac{1}{|c_n|^{\lambda-\epsilon}}$ diverges.

The μ-index of $\theta_1(x)=\Sigma_{n=1}^{\infty}u_n x^n$ is, if it exists, the number $\mu\geqq 0$, such that to every positive ϵ

(a) $|u_n|<\frac{1}{n!^{\mu-\epsilon}}$ for all $n>m_\epsilon$,

(b) $|u_n|>\frac{1}{n!^{\mu+\epsilon}}$ for some arbitrarily large n.

* The deduction of this functional equation is due to C. L. Bouton.

† Vivanti-Gutzmer, *Eindeutige analytische Funktionen*, pp. 228–230.

The ν-index of $\theta_1(x)$ is, if it exists, the number $\nu \geqq 0$, such that to every positive ϵ*

(*a*) $|\theta_1(x)| < e^{|x|^{\nu+\epsilon}}$ for all $|x| > r_\epsilon$,

(*b*) $|\theta_1(x)| > e^{|x|^{\nu-\epsilon}}$ for some arbitrarily large x.

In case an index does not exist, it is said to be infinite. The following theorems are dependent on the definition of the μ-index:

THEOREM 1. *If $\theta_1(x)$ has μ-index infinite, $\theta(x)$ is an integral function with μ-index infinite, and conversely.*

THEOREM 2. *If $\theta_1(x)$ has μ-index equal to μ_1, then*

(*a*) *if $\mu_1 > 1$, $\theta(x)$ is an integral function, with μ-index equal to $\mu_1 - 1 > 0$;*

(*b*) *if $\mu_1 < 1$, $\theta(x)$ is divergent and*

$$|n!\, u_n| < n!^{1-\mu_1+\epsilon},\ n > m_\epsilon,$$

$$|n!\, u_n| > n!^{1-\mu_1-\epsilon},\ \textit{for some arbitrarily large } n, \textit{ and conversely.}$$

If $\mu_1 = 1$, then $|n!\, u_n| < n!^{\epsilon}$ for all $n > m_\epsilon$, $|n!\, u_n| > n!^{-\epsilon}$ for some arbitrarily large n, and $\theta(x)$ may be an integral function with μ-index equal to zero, or may be convergent for $|x| < \rho$, or divergent. To distinguish these three cases, we introduce, in connection with the ν-index, which here is unity, since $\mu\nu = 1$,† a subindex γ,‡ and thus obtain a further classification of integral functions with μ-index unity.

For $\theta_1(x)$ with $\mu = \nu = 1$,

(*a*) the γ-subindex is infinite, if $\begin{cases} |\theta_1(x)| < e^{|x|^{1+\epsilon}} & \text{for all } |x| > r_\epsilon, \\ |\theta_1(x)| > e^{\frac{|x|}{\epsilon}} & \text{for some arbitrarily large } x; \end{cases}$

(*b*) the γ-subindex is $\gamma \neq 0$, if $\begin{cases} |\theta_1(x)| < e^{(\gamma+\epsilon)|x|} & \text{for all } |x| > r_\epsilon, \\ |\theta_1(x)| > e^{(\gamma-\epsilon)|x|} & \text{for some arbitrarily large } x; \end{cases}$

(*c*) the γ-subindex is zero, if $\begin{cases} |\theta_1(x)| < e^{\epsilon|x|} & \text{for all } |x| > r_\epsilon, \\ |\theta_1(x)| > e^{|x|^{1-\epsilon}} & \text{for some arbitrarily large } x. \end{cases}$

Pringsheim§ shows that these three pairs of inequalities on $|\theta_1(x)|$ lead respectively to the following three pairs of inequalities on $|u_n|$ and conversely:

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* Pringsheim, *Ganze Funktionen*, Mathematische Annalen, vol. 58 (1904), p. 262.

† Pringsheim, loc. cit., p. 279; Vivanti-Gutzmer, loc. cit, p. 267

‡ Pringsheim, loc. cit., p. 264.

§ Pringsheim, loc. cit., pp. 278–279.

$$(a)\qquad \begin{cases} |u_n| < \dfrac{1}{n!^{1-\epsilon}} \text{ for all } n > m_\epsilon, \\ |u_n| > \dfrac{1}{n!^{\epsilon^n}} \text{ for some arbitrarily large } n; \end{cases}$$

$$(b)\qquad \begin{cases} |u_n| < \dfrac{(\gamma+\epsilon)^n}{n!} \text{ for all } n > m_\epsilon, \\ |u_n| > \dfrac{(\gamma-\epsilon)^n}{n!} \text{ for some arbitrarily large } n; \end{cases}$$

$$(c)\qquad \begin{cases} |u_n| < \dfrac{\epsilon^n}{n!} \text{ for all } n > m_\epsilon, \\ |u_n| > \dfrac{1}{n!^{1+\epsilon}} \text{ for some arbitrarily large } n. \end{cases}$$

The corresponding inequalities on the coefficients of $\theta(x)$ are:

$$(a)\qquad \begin{cases} |n!\,u_n| < n!^{\epsilon} \text{ for all } n > m_\epsilon, \\ |n!\,u_n| > \epsilon^{-n} \text{ for some arbitrarily large } n; \end{cases}$$

$$(b)\qquad \begin{cases} |n!\,u_n| < (\gamma+\epsilon)^n \text{ for all } n > m_\epsilon, \\ |n!\,u_n| > (\gamma-\epsilon)^n \text{ for some arbitrarily large } n; \end{cases}$$

$$(c)\qquad \begin{cases} |n!\,u_n| < \epsilon^n \text{ for all } n > m_\epsilon, \\ |n!\,u_n| > n!^{-\epsilon} \text{ for some arbitrarily large } n. \end{cases}$$

Hence we conclude the following theorem:

THEOREM 3. *If $\theta_1(x)$ has μ-index equal to unity, then*

(a) if γ is infinite, $\theta(x)$ diverges;

(b) if γ is finite, not zero, $\theta(x)$ converges for $|x| < \dfrac{1}{\gamma}$;

(c) if γ is zero, $\theta(x)$ is an integral function with μ-index equal to zero, and conversely.

We are now in a position to discuss the conditions for a solution of the equation (14).

THEOREM 4. *A necessary and sufficient condition that there exist a convergent $\psi(x)$ satisfying* (14) *and hence that there exist an analytic curve $y = \psi(x)$ invariant under* (13) *is that $\varphi_1(x)$ contain among its zeros the zeros: $2n\pi i$, $n = 0, \pm 1, \pm 2, \ldots$, of $f_1(x) = e^x - 1$. The radius of convergence of $\psi(x)$, when $\psi(x)$ converges, is at least as great as that of $\varphi(x)$.*

The condition is necessary, for if $\psi(x)$ is convergent, $\psi_1(x)$ is an integral function and $\varphi_1(x)$ must have among its zeros the zeros of $f_1(x)$.

It remains to show the condition sufficient. The function $f_1(x)$ has the zeros $2n\pi i$, its λ-, μ-, ν- indices and γ-subindex are all unity. Since $\varphi(x)$ converges, μ_{φ_1} is, by Theorems 1 and 2, greater than one, or, by the Theorem 3, equal to one with γ_{φ_1} finite. Suppose $\mu_{\varphi_1} > 1$; then since $\nu = 1/\mu$ and $\lambda \geqq \nu^*$ it follows that $\lambda_{\varphi_1} < 1$; but in order that $\varphi_1(x)$ have among its zeros the values $2n\pi i$, $\lambda_{\varphi_1} \geqq 1$ and hence a contradiction. We thus have

$$\mu_{\varphi_1} = 1, \nu_{\varphi_1} = 1 \text{ and } \gamma_{\varphi_1} \text{ finite}, \lambda_{\varphi_1} = 1.$$

We next prove the lemma:

LEMMA: *If for a given γ, ν, $|\psi_1(x)| > e^{\gamma|x|^\nu}$ for some arbitrarily large x, then for any positive ϵ, there exists some arbitrarily large x for which $|\varphi_1(x)| > e^{(\gamma-\epsilon)|x|^\nu}$.*

With the points $x = 2n\pi i$ as centers and radius $\eta < \pi/2$ draw circles. Denote by S_η that part of the plane which remains after the interiors of the circles have been removed. In S_η, $|e^x - 1| \geqq (1 - 1/e^\eta)$, for in the fundamental region: $-\pi \leqq x_2 < \pi$, $x = x_1 + ix_2$, exclusive of the interior of the circle $|x| = \eta$, $e^x - 1$ has its minimum in $x = -\eta$.

Denote by X those of the given x which are not in S_η. We replace each X by that x in S_η which lies on the circle, center in X, radius 2η and in which $|\psi_1(x)|$ for the circle takes on its maximum value. Then by Cauchy's Abschätzung† we have

$|\psi_1(x)| > |\psi_1(X)|$, that is, $|\psi_1(x)| > e^{\gamma|X|^\nu}$. Since $|X| - |x| \geqq -2\eta$, we have

$$|\psi_1(x)| > e^{\gamma|x|^\nu\left(1-\frac{2\eta}{|x|}\right)^\nu}.$$

For each $\epsilon_1 > 0$ there exists r_{ϵ_1} such that $\left(1-\frac{2\eta}{|x|}\right)^\nu > 1 - \frac{\epsilon_1}{\gamma}$, for $|x| > r_{\epsilon_1}$.

Hence the relation

$$|\psi_1(x)| > e^{(\gamma-\epsilon_1)|x|^\nu}$$

holds for those $|x| > r_{\epsilon_1}$ in S_η which replace the given X not in S_η; it certainly holds for those of the given x in S_η and therefore holds for infinitely many arbitrarily large x, $|x| > r_{\epsilon_1}$. We accordingly have $|\varphi_1(x)| > e^{(\gamma-\epsilon_1)|x|^\nu}(1 - 1/e^\eta)$ for some arbitrarily large x, $|x| > r_{\epsilon_1}$. Given a positive ϵ, take $\epsilon_1 < \epsilon$; then there exists $r_\epsilon \geqq r_{\epsilon_1}$ such that $(1 - 1/e^\eta) > e^{(\epsilon_1-\epsilon)|x|^\nu}$ for $|x| > r_\epsilon$ and hence $|\varphi_1(x)| > e^{(\gamma-\epsilon)|x|^\nu}$ for some arbitrarily large x, $|x| > r_\epsilon$.

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* Vivanti-Gutzmer, loc. cit., p. 238.

† Osgood, *Funktionentheorie*, 1st. edition, p. 255.

We conclude at once that ν_{ψ_1} is finite, for if ν_{ψ_1} were infinite then, by the lemma, ν_{φ_1} would be infinite, whereas it is actually unity. Also by the lemma $\nu_{\psi_1} \leqq \nu_{\varphi_1}$, that is, $\nu_{\psi_1} \leqq 1$, $\mu_{\psi_1} \geqq 1$.

If $\mu_{\psi_1} > 1$, by Theorem 2, $\psi(x)$ converges for all x and the sufficiency of the condition in Theorem 4 is in this case proved.

If $\mu_{\psi_1} = 1$, then $\nu_{\psi_1} = \nu_{\varphi_1} = 1$ and γ_{ψ_1} is finite, for, if ν_{ψ_1} were infinite, the lemma applied for $\nu = 1$ yields γ_{φ_1} infinite which means that $\varphi(x)$ diverges. Since $\nu_{\psi_1} = 1$ and γ_{ψ_1} is finite, by Theorem 3, $\psi(x)$ converges and the sufficiency proof is complete.

An inequality which $\rho \neq 0$, the radius of convergence of the ψ–series, must satisfy, may be easily determined. From the lemma when $\nu_{\psi_1} = 1$, $\gamma_{\varphi_1} \geqq \gamma_{\psi_1}$; also $|\varphi_1(x)| < e^{(1+\epsilon)|x|} e^{(\gamma_{\psi_1}+\epsilon)|x|}$, $|x| > r_\epsilon$. Therefore $\gamma_{\psi_1} \leqq 1 + \gamma_{\varphi_1}$, or $\gamma_{\varphi_1} \geqq \gamma_{\psi_1} \geqq \gamma_{\varphi_1} - 1$. Since $\gamma = 1/\rho$, this gives

$$\rho_\varphi \leqq \rho_\psi \leqq \frac{\rho_\varphi}{1-\rho_\varphi}.$$

When $\nu_{\psi_1} < 1$, $\mu_{\psi_1} > 1$ and ρ_ψ is infinite, while $\rho_\varphi \geqq 1$ for $\gamma_{\varphi_1} \leqq 1$. Hence always

$$\rho_\varphi \leqq \rho_\psi \text{ when } \rho_\psi \neq 0.$$

According to Theorem 4 the transformation (13) when $\varphi(x)$ is a polynomial beginning with terms of at least the third degree has no invariant analytic curve $y = \psi(x)$ corresponding to the simple root of the cubic (6), though there is obviously a formal ψ-series. The curve $y = cx^2$ is left invariant when $\varphi(x) = \sum_{n=3}^{x}(n-1)x^n$, while the curve $y = x^2/(1-Ax)^2$ is left invariant under (13) when $\varphi_1(x) = x^2e^{Ax}(e^x - 1)$.

That the number of invariant analytic curves may be infinite is seen from the transformation:

$$x_1 = x + x^2, \quad y_1 = y + 2xy + x^2y,$$

which leaves invariant the ∞^1 curves $y = l_2x^2$, l_2 arbitrary; also from the transformation:

$$x_1 = x[1 + \prod_{j=1}^{m}(y-\varphi_j(x))],\ y_1 = y[1 + \prod_{j=1}^{m}(y-\varphi_j(x))]^k,$$

where $\varphi_j(x)$ is analytic at $x=0$, $\varphi(0) = 0$, m,k are positive integers, which leaves invariant the ∞^1 analytic curves $y = l_kx^k$, l_k arbitrary and also the m other curves $y = \varphi_j(x)$ $(j = 1, 2, \ldots, m)$. On the other hand the transformation

$$x_1 = x + y^2,\ y_1 = y + x^3,$$

possesses not even a formally invariant curve.

The results obtained in this paragraph show that we can not in general hope to predict the number of power series developments formally invariant under a given transformation, nor, when they exist, can we expect them always to be convergent.

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DIE ZERLEGUNG VON PRIMZAHLEN IN ALGEBRAISCHEN ZAHLKÖRPERN*

VON

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In den folgenden Zeilen wird gezeigt, dass das Problem der Zerlegung einer Primzahl in einem algebraischen Körper identisch ist mit einem Problem der Theorie der linearen Substitutionen in einem Galois-Feld,† das folgendermassen formuliert werden kann:

Gegeben ist eine ganzzahlige quadratische Matrix M. Nimmt man die Reste modulo einer Primzahl p, so erhält man die Matrix einer linearen Substitution in einem $GF(p)$. Ihre Ordnung hat die Gestalt $p^s u$, wobei u prim ist zu p und eine gewisse Zahl $p^f - 1$ teilt. Hier ist f von der Gestalt $f = n_1 n_2 \ldots n_r$, wobei $n_1 + n_2 + \cdots + n_r \leqq n$ und n ist der Grad der Substitution. Wir nennen die niedrigste Zahl f, für welche $p^f - 1 \equiv 0 \pmod{u}$, die *Exponentialzahl* von M mod p und zeigen in § 1, dass sie den *Grad* der in p aufgehenden Primideale darstellt für denjenigen algebraischen Zahlkörper, der durch die Wurzeln der characteristischen Gleichung dieser Matrix bestimmt ist. Hierbei müssen die Primteiler der Discriminante dieser Gleichung ausgenommen werden, deren Untersuchung in dieser Arbeit nicht durchgeführt werden soll, da sie weitere Hilfsmittel erfordert. In § 2 werden die bekannten Criterien für die Zerlegung in zyklischen und relativ-zyklischen Körpern hergeleitet. Hierbei tritt klar zutage, dass das Zerlegungsproblem für den Fall von Körpern, die nicht durch Wurzelzeichen herstellbar sind, mit den bisher benutzten Methoden nicht mehr zu behandeln ist. Damit man aber wenigstens in der Lage ist, für die niedrigsten Primzahlen den Grad zu bestimmen, so wird in §3 eine rekurrente Reihe benutzt, welche wohl schon öfters in der Literatur aufgetreten ist, ohne dass ihre enge Beziehung zum Zerlegungsproblem bisher erkannt worden wäre.

1. Die Zerlegung von Primzahlen in Zahlkörpern

Gegeben sei die Gleichung:

$$x^n - (a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n) = 0$$

mit ganzen rationalen Koeffizienten. Wir fragen nach der Zerlegung der Primzahl p in dem durch die Wurzeln dieser Gleichung erzeugten Galois' schen Körper.

* Presented to the Society, April 14, 1922.

† L. E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, Leipzig, 1901.

Wenn ein Primidealteiler $\mathfrak{p}$ von p vom Grade f ist, so bilden bekanntlich die Reste mod $\mathfrak{p}$ ein $GF(p^f)$. Es gilt nun der

SATZ 1. *$GF(p^f)$ ist das GF mit niedrigstem Exponenten f, in welchem bei gegebenem p die linke Seite der Gleichung in ihre Linearfaktoren zerfällt. Hierbei muss p prim sein zur Diskriminante der Gleichung.*

Beweis: Zunächst ist klar, dass die Funktion im $GF(p^f)$ in ihre Linearfaktoren zerfällt, denn jede der Wurzeln $\alpha_1, \alpha_2, \ldots, \alpha_n$ gibt mod $\mathfrak{p}$ ein bestimmtes Element des $GF(p^f)$. Es muss also nur noch gezeigt werden, dass diese Zerlegung in keinem GF mit niedrigerem Exponenten, der übrigens ein Teiler von f sein müsste, möglich ist. Angenommen nämlich, sie sei bereits im $GF(p^{f_1})$ möglich, so müssten die Reste der Wurzeln $\alpha_1, \alpha_2, \cdots, \alpha_n$ mod $\mathfrak{p}$ wegen des Satzes von der eindeutigen Zerlegung der Funktionen in einem GF, durch Addition, Subtraktion und Multiplikation gerade das $GF(p^{f_1})$ erzeugen. Nun entsteht aber durch die genannten drei Operationen aus den Wurzeln selbst ein Ring, dessen Diskriminante nur diejenigen Primteiler enthält, die bereits in der Diskriminante der Gleichung aufgehen. Die Basis des Ringes geht daher aus einer Körperbasis durch eine Substitution hervor, deren Determinante auch nur solche Primteiler enthält, und infolgedessen zu p prim ist. Hieraus folgt ohne weiteres, dass das Restesystem mod $\mathfrak{p}$ im Körper und im Ring identisch ist, d. h. dass $f_1 = f$.

SATZ 2. *Es sei u die niedrigste ganze positive Zahl, für welche die n Gleichungen gelten: $\alpha_i^u \equiv 1$ oder $0 \pmod{\mathfrak{p}}$ und f die kleinste Zahl, für welche $p^f \equiv 1 \pmod{u}$, dann ist f der Grad von $\mathfrak{p}$.*

Beweis: Ist f der Grad von $\mathfrak{p}$, so bilden die zu p primen Reste eine zyklische Gruppe von der Ordnung $p^f - 1$. Wenn nun die Zahl u bereits einer Kongruenz mit niedrigerem Exponenten genügte, der alsdann bekanntlich ein Teiler von f wäre, so würden die Reste der Wurzeln mod $\mathfrak{p}$ nicht das ganze $GF(p^f)$ erzeugen, was dem Satz 1 widerspricht.

SATZ 3. *Ist $(a_i^{(k)})$ $(i, k = 1, 2, \cdots, n)$ eine ganzzahlige Matrix und ist f ihre Exponentialzahl* mod p *(vgl. die Einleitung), so zerfällt p in dem durch die Wurzeln der characteristischen Gleichung gebildeten Galois'schen Körper in Primideale vom Grade f. Immer sind ausgenommen die Teiler der Diskriminante der Gleichung.*

Beweis: Setzt man die Matrix (a_{ik}) h-mal mit sich selbst zusammen, so sind die Wurzeln der charakteristischen Gleichung die h-ten Potenzen von $\alpha_1, \alpha_2, \ldots, \alpha_n$. Bildet man daher insbesondere die $(p^f - 1)$-ste Potenz, so sind die Wurzeln der so entstehenden Matrix $\equiv 0$ oder $1 \pmod{\mathfrak{p}}$. Die Ordnung einer solchen Matrix ist aber mod p immer gleich einer Potenz von p. Da die auftretenden Matrizen sämtlich ganze rationale Koeffizienten besitzen, so darf man den Modul $\mathfrak{p}$ durch p ersetzen.

Die Einschränkung, der die Moduln p unterliegen, kann dadurch aufgehoben werden, dass an die Stelle einer beliebigen Gleichung die Fundamentalgleichung

des Körpers gesetzt wird. Ferner ist zu bemerken, dass die Gleichung nicht als irreduzibel vorausgesetzt werden muss, was sich z. B. bei den Kreiskörpern vorteilhaft bemerkbar macht.

Jede Gleichung $x^n = a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n$ ist charakteristische Gleichung einer Matrix, nämlich der folgenden:

$$\left|\left|\begin{array}{l} 0, 0, 0, \ldots, 0, a_n \\ 1, 0, 0, \ldots, 0, a_{n-1} \\ 0, 1, 0, \ldots, 0, a_{n-2} \\ 0, 0, 1, \ldots, 0, a_{n-3} \\ \ldots\ldots\ldots\ldots\ldots \\ 0, 0, 0, \ldots\ldots, a_1 \end{array}\right|\right| = M$$

und in dieser Gestalt werden wir das Kriterium des Satzes im folgenden benutzen.

2. Die Kriterien für die Kreiskörper und die relativ-zyklischen Körper

1. **Kreiskörper.** Die Gleichung lautet hier:

$$x^n = 1,$$

wobei n irgend eine ganze positive Zahl bedeutet. Bildet man die zugehörige Matrix, so findet man, dass ihre Ordnung modulo jeder Primzahl gleich n ist, denn sie ist gleichbedeutend mit einer zyklischen Permutation. Ist daher p kein Teiler von n, so folgt das bekannte Kriterium:

Der Grad f eines Primteilers von p im Körper der n-ten Einheitswurzeln ist gleich dem Exponenten der niedrigsten Potenz von p, welche der Gleichung genügt:

$$p^f \equiv 1 \pmod{n}.$$

2. **Reine Gleichungen.** Gegeben ist die Gleichung:

$$x^l = a,$$

wobei l eine Primzahl bedeutet und a keine l-te Potenz einer rationalen Zahl ist. Wenn $a \pmod{p}$ zum Exponenten q gehört: $a^q \equiv 1 \pmod{p}$, so wird $u = l \cdot q$. Die Primteiler von p sind dann und nur dann vom ersten Grad, wenn $p \equiv 1 \pmod{lq}$, d. h. wenn die Gleichung $x^l \equiv a \pmod{p}$ lösbar ist. In diesem Fall sagt man: *a ist l-ter Potenzrest* mod p.

Ferner gilt der

Satz 4. *Der Grad f der Primidealteiler von p in einem aus zwei Galois'schen Körpern zusammengesetzten Körper ist gleich dem kleinsten gemeinschaftlichen Vielfachen der entsprechenden Grade in den beiden Teilkörpern.*

Beweis: In dem $GF(p^f)$ werden die Fundamentalgleichungen beider Teilkörper zerlegbar in ihre Linearfaktoren. Ausgenommen sind auch hier die Diskriminantenteiler.

Die bisherigen Ueberlegungen lassen sich ohne wesentliche Aenderungen auf Gleichungen in einem beliebigen algebraischen Grundkörper k übertragen. Ist $\mathfrak{p}$ ein Primideal vom Grade f_1, so ergibt die Gleichung mod $\mathfrak{p}$ eine Matrix im $GF(p^{f_1})$. Ist f ihre Exponentialzahl, so zerfällt $\mathfrak{p}$ in dem durch die Wurzeln der Gleichung bestimmten relativ zu k Galois'schen Zahlkörper in Primideale vom absoluten Grade f.

3. **Die relativ zyklischen Zahlkörper.** Gegeben ist ein Zahlkörper k und eine Gleichung $x^l = \alpha$, wobei α in k liegt und keine l-te Potenz einer Zahl aus k ist. Der durch diese Gleichung bestimmte Zahlkörper enthält den Körper der l-ten Einheitswurzeln. Adjungiert man diesen zu k, so reduziert sich die Gruppe der Gleichung auf die zyklische Gruppe von der Ordnung l. Ist nun $\mathfrak{p}_1$ ein Primideal in k, so wird seine Zerlegung nach Adjunktion des Kreiskörpers durch den Satz 4 bestimmt. $\mathfrak{p}$ sei ein Primteiler von $\mathfrak{p}_1$. Die Zerlegung von $\mathfrak{p}$ im vollen Körper erfolgt nun genau nach 2 und es gilt der Satz:

Satz 5. *Das Primideal $\mathfrak{p}$ zerfällt in l Faktoren oder es bleibt Primideal, je nachdem α l-ter Potenzrest ist* (mod $\mathfrak{p}$) *oder nicht im Körper, dem $\mathfrak{p}$ angehört.*

Hiermit sind die bisher bekannten Regeln für die Zerlegung von Zahlen resp. Idealen in Zahlkörpern, abgesehen von den Diskriminantenteilern, hergeleitet. Man sieht, dass sie nur für auflösbare Körper resp. Relativkörper gelten, infolge der besonders einfachen Gestalt der zugehörigen Matrix, die eine monomiale wird. Nimmt man allgemeinere Gleichungen, etwa die im nächsten Paragraphen zu besprechende Gleichung: $x^5 = x + 1$, so gelangt man auf Matrizen, für die eine ähnlich einfache Regel zur Bestimmung der Exponentialzahl nicht mehr zu erwarten ist, und es muss die Frage noch offen gelassen werden, ob überhaupt ein einfaches Zerlegungsgesetz existiert oder ob die bisherige Formulierung das Schlussresultat darstellt. Jedenfalls ist die Zurückführung auf Matrizen in Diagonalform, d. h. auf die Lehre von den Potenzresten nur im Falle auflösbarer Körper möglich, genau wie auch nur in diesem Fall das algebraische Gleichungsproblem auf ein Formenproblem l-ten Grades reduzierbar ist.*

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* Als Literatur über diese Fragen seien folgende Abhandlungen angemerkt:

D. Hilbert, *Die Theorie der algebraischen Zahlkörper*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 4, S. 189, S. 333 und S. 398; *Ueber die Theorie der relativ-abelschen Zahlkörper*, Göttinger Nachrichten, 1898, S. 370–399.

Ueber den Zusammenhang mit den Reziprozitätsgesetzen vgl. R. Fueter, *Die Klassenkörper der komplexen Multiplikation und ihr Einfluss auf die Entwicklung der Zahlentheorie*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 20.

3. Ein Algorithmus zur Berechnung des Grades eines Primideales

Da sich im allgemeinen Fall kein einfaches Gesetz für den Grad eines Primideales angeben lässt, so ist es von Wichtigkeit, ein Verfahren zu kennen, das wenigstens für kleine Primzahlen zum Ziele führt, ohne allzuweitläufige Rechnungen. Es sei wiederum die Gleichung gegeben:

$$x^n = a_1 x^{n-1} + \cdots + a_n.$$

Wir betrachten diejenigen Primzahlen, welche zur Diskriminante sowie zu a_n prim sind. Aus den Koeffizienten dieser Gleichung kann man eine Rekursionsformel herleiten, welche aus der Entwicklung des reziproken Wertes von $x^n - a_1 x^{n-1} - \cdots - a_n$ in eine Potenzreihe bekannt ist. Hiernach bildet man aus n aufeinanderfolgenden Zahlen $y_1, y_2, \ldots, y_n$ die nächstfolgende durch die Formel

$$y_{n+1} = a_n y_1 + a_{n-1} y_2 + \cdots + a_1 y_n.$$

Die ersten n Zahlen können beliebig gewählt werden. Wir wählen sie aber in bestimmter Weise, indem wir setzen: $y_1 = 0$, $y_2 = 0$, $\ldots$, $y_{n-1} = 0$, $y_n = 1$. Indem man die Rekursionsformel successive anwendet, erhält man eine unbegrenzte Reihe ganzer Zahlen. Diese Zahlen reduziere man mod p. Sie wird alsdann zu einer periodischen Reihe der Reste und wir bezeichnen die Anzahl der Terme in einer Periode mit u, alsdann gilt der

Satz 6. *Der Grad der in p aufgehenden Primideale ist die kleinste Zahl f, für welche die Kongruenz gilt:* $p^f \equiv 1 \pmod{u}$.

Bevor wir diesen Satz beweisen, sei ein Beispiel dafür gegeben. Wir wählen die Gleichung: $x^5 = x + 1$. Ihre Diskriminante ist $2869 = 19 \cdot 151$. Als Primzahl nehmen wir 2. Man findet sofort folgende Reihe:

$$0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, \ldots$$

und es erweist sich sonach $u = 21$, und also $f = 6$. Für $p = 3$ findet man: $u = 121$ und also $f = 5$, da $3^5 = 243$. Aus diesen beiden Resultaten folgt übrigens, dass die Gruppe der Gleichung die symmetrische von der Ordnung 120 ist, denn man zeigt in der Zahlentheorie, dass der Grad eines Primideales gleich der Ordnung einer zyklischen Untergruppe der Körpergruppe ist. Nun ist offenbar die symmetrische Gruppe die einzige Permutationsgruppe von 5 Variablen, welche Substitutionen von der Ordnung 5 und 6 enthält.

Um den Satz 6 zu beweisen, stellen wir folgende Ueberlegungen an: Reduziert man die Gleichung mod p, so zerfällt sie in einem gewissen $GF(p^f)$ in ihre Linearfaktoren. Ihre Wurzeln sind wegen unserer Voraussetzung über p unter einander und von 0 verschieden. Daher lässt sich ihre Matrix M (vergl. § 1 am Ende) auf die Diagonalform bringen im $GF(p^f)$. Hieraus folgt, dass die

Ordnung von $M \pmod p$ ein Teiler von $p^f - 1$ ist. f ist der Grad der in p aufgehenden Primideale. Nun gehen wir über zu der rekurrenten Reihe $y_1, y_2, \ldots$ und betrachten irgendwelche $2n - 1$ aufeinanderfolgende Glieder: $y_{i+1}, y_{i+2}, \ldots, y_{i+2n-1}$. Aus diesen bilden wir die folgende (Hankelsche) Matrix:

$$M_i = \begin{Vmatrix} y_{i+1} & y_{i+2} & \cdots & y_{i+n} \\ y_{i+2} & y_{i+3} & \cdots & y_{i+n+1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{i+n} & y_{i+n+1} & \cdots & y_{i+2n-1} \end{Vmatrix}.$$

Diese Matrizen genügen folgender Gleichung: $M_i M = M_{i+1}$, wobei Zeilen mit Kolonnen zusammenzusetzen sind. Aus der besonderen Wahl von $y_1, \ldots, y_n$ folgt, dass die Determinante von M_0 gleich ± 1 ist. Da diejenige von M gleich a_n wird, so sind die Determinanten sämtlicher Matrizen $M_i \not\equiv 0 \pmod p$. Wenn nun die rekurrente Reihe mod p periodisch mit u Gliedern ist, so wird $M_u \equiv M_0$ und also $M_0 M^u \equiv M_0$. Hier kann man mit M_0 kürzen und man findet: $M^u \equiv E$, wo E die Einheitsmatrix darstellt. Umgekehrt folgt aus $M^u \equiv E$, dass die Reihe eine Periode von u Gliedern aufweist. Die Anzahl u der Glieder in einer Periode ist also gleich der Ordnung der Matrix $M \pmod p$, und hieraus folgt die Giltigkeit des Satzes 6.

Durch diesen Satz ist gleichzeitig die zahlentheoretische Natur der rekurrenten Reihen vollständig aufgeklärt. Man sieht, dass auch dieses Problem identisch ist mit der Frage nach dem Grad der in p aufgehenden Primideale in einem Galois'schen Zahlkörper. Was nun das Verhalten der Diskriminantenteiler betrifft, so ist zu bemerken, dass auch diese Untersuchung auf Probleme der Galois'schen Felder führt und Anlass zu interessanten Problemen der Gruppentheorie bietet.*

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* Vgl. D. Hilbert, *Die Theorie der algebraischen Zahlkörper*, Jahresbericht der Deutschen Mathematiker-Vereinigung, Bd. 4, S. 250 ff.
A. Speiser, *Die Zerlegungsgruppe*, Journal für Mathematik, Bd. 149, S. 174.

ZURICH,
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THE ELLIPTIC MODULAR FUNCTIONS ASSOCIATED WITH THE ELLIPTIC NORM CURVE E^7*

BY

ROSCOE WOODS

INTRODUCTION.

The elliptic norm curve E^n in space S_{n-1} admits a group G_{2n^2} of collineations and there is a single infinity of such curves which admit the same group. A particular E^n of the family is distinguished by the coördinates of a point on a modular curve, the ratios of these coördinates being elliptic modular functions defined by the modular group congruent to identity (mod n). In the group G_{2n^2} there are certain involutory collineations with two fixed spaces. If E^n is projected from one fixed space upon the other, a family of rational curves C^m mapping the family of E^n's is obtained. The quadratic irrationality separating involutory pairs on E^n involves the coördinates of a point on the modular curve and the parameter t on a member of the family C^m.

Miss B. I. Miller† has discussed the elliptic norm curves for which $n = 3, 4, 5$. In these cases the genus of the modular group is zero and a point of the modular curve can be denoted by a value of the binary parameter τ. The irrationality separating involutory pairs on E^n was used by her to define an elliptic parameter

$$u = \int \frac{(tdt)}{\sqrt{(t\tau)\alpha_\tau^{r-3}\alpha_t^3}}$$

where α_τ^r is the tetrahedral, octahedral, or icosahedral form. This form of u is invariant under all the cogredient tranformations of t and τ which leave α_τ^r unaltered.

The cases considered by Dr. Miller are relatively simple, due to the fact that the genus of the modular group is zero. In this paper, the case $n = 7$ for which the genus is 3, one which is fairly typical of the general case, is subjected to a

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* Presented to the Society, April 14, 1922.

† See these Transactions, vol. 17 (1916), p. 259.

similar investigation. Many of the results may be extended to the case where n is any prime number and in some features to the case where n is any odd number. By methods of geometry and group theory, we derive in this discussion the well known elliptic modular functions attached to this group as well as some new ones and obtain a number of their algebraic properties.* This treatment suggests a number of "root functions," i. e., square roots of modular functions which are themselves uniform.

In §I, the groups and subgroups associated with the E^7 are discussed and thrown into a canonical form. The equations of the transformation from S_6 to the fixed spaces S_2, S_3, and the equations of the groups of transformations in these spaces are derived. These have been found without the aid of function theory and have been checked with Klein's results in Klein-Fricke's *Elliptische Modulfunktionen*. In §II, a single Kleinian form† is derived which furnishes the fourteen linearly independent quadrics whose complete intersection is E^7. From this form in §III the fundamental elliptic modular functions $t_1 : t_2 : t_4$ are determined. Also the families C^2, C^3 of rational curves in S_2 and S_3 are found. In §IV, the loci in S_3 are discussed. The paper closes with a parametric representation of E^7.

I. The groups connected with E^7

1. **The group $G_{2\cdot 7^2}$ of collineations of E^7 into itself.** The homogeneous coördinates of a point of the elliptic norm curve E^7 are $x_0 : x_1 : \cdots : x_6 = 1 : p(u) : p'(u) : \cdots : p^5(u)$. As u runs over the period parallelogram ω_1, ω_2 the E^7 is obtained in a six-dimensional space S_6. It is known‡ that the only birational transformations of the general elliptic curve into itself are given by $u' = \pm u + b$, where b is any constant.§ From the parametric representation of the E^7 as set forth above, it is evident that seven points of the E^7 on a hyperplane section are characterized by the fact that the sum of their parameters is congruent to zero (mod ω_1, ω_2) and conversely. In view of this, all transformations for which $7b \equiv 0 \pmod{\omega_1, \omega_2}$ are collineations. This congruence has three irreducible solutions

$$b = 0, \quad b = \omega_1/7, \quad b = \omega_2/7. \tag{1}$$

* In the case $n = 4$, Miss Miller has expressed the opinion that the properties of the elliptic integral associated with E^4 and the Dyck quartic should apply to Klein's quartic which occurs in this case. This has not been verified.

† By a Kleinian form is meant a form in several variables invariant under isomorphic linear groups on these variables.

‡ Appell-Goursat, *Fonctions Algébriques*, p. 474.

§ Segre, Mathematische Annalen, vol. 27(1887), p. 296.

Klein-Fricke, *Theorie der elliptischen Modulfunktionen*, vol. 2, p. 241. Hereafter the initials K. F. will be used to refer to this work.

These furnish the substitutions

$$
\begin{aligned}
&S_{01} : u' = u + \omega_{01}, \\
&S_{10} : u' = u + \omega_{10}, \qquad \omega_{ij} = i\omega_1/7 + j\omega_2/7, \\
&V \ : u' = -u, \qquad\qquad (i, j = 0, 1, \ldots, 6).
\end{aligned} \tag{2}
$$

S_{01} and S_{10} are collineations of period seven and generate a group G_{7^2} which is abelian in its elements. V is a collineation of period two which adjoined to G_{7^2} generates a group $G_{2\cdot 7^2}$. This group $G_{2\cdot 7^2}$ of collineations contains *all* the collineations of the general E^7 into itself.

The G_{7^2} in the $G_{2\cdot 7^2}$ contains 8 cyclic G_7's and no other subgroups. These are denoted by $G_\infty, G_1, \ldots, G_6$ where G_∞ is generated by S_{01} and G_i by $S_{10}S_{01}^i$ $(i = 0, 1, \ldots, 6)$. The elements of $G_{2\cdot 7^2}$ not in G_{7^2} are of the form

$$
V_{ij} : u' = -u + \omega_{ij} \qquad (i, j = 0, 1, \ldots, 6), \tag{3}
$$

and are of period two. The V_{ij} form a conjugate set. Any cyclic G_7 with one involution generates a dihedral $G_{2\cdot 7}$ which contains seven involutions. Hence there are 56 dihedral $G_{2\cdot 7}$'s. These with the cyclic G_2's complete the subgroups of $G_{2\cdot 7^2}$. The relations satisfied by the generators of $G_{2\cdot 7^2}$ are

$$
\begin{aligned}
&S_{01}^7 = S_{10}^7 = 1, \\
&S_{01}S_{10} = S_{10}S_{01}, \quad VS_{10} = S_{10}^{-1}V, \quad VS_{01} = S_{01}^{-1}V.
\end{aligned} \tag{4}
$$

2. **The fixed heptahedra of the 8 cyclic G_7's.** The condition that a hyperplane section touch the E^7 in seven coincident points is given by

$$
7u \equiv 0 \pmod{\omega_1, \omega_2}. \tag{5}
$$

The irreducible solutions of this congruence furnish the 49 parameters ω_{ij} of the singular points. Under G_∞ the 49 points ω_{ij} separate into 7 sets of seven conjugate points such that each set is on a hyperplane. Such a set of seven hyperplanes will be called a *heptahedron*. Since there are 8 cyclic G_7's, there are 8 heptahedra which will be designated by $H_\infty, H_0, \ldots, H_6$.*

The 49 singular points are now arranged in a matrix (using only the subscripts) in such a way that the rows furnish the 7 sets of conjugate points which determine the 7 hyperplanes of H_∞, while the columns furnish the 7 hyperplanes of H_0.

* The reason for calling one heptahedron H_∞ will appear later. These heptahedra can be determined from the resolvent equation of the 8th degree associated with the Galois problem of degree 168. Compare K. F., vol. 1, p. 732.

$$(6)\qquad \left\| \begin{array}{ccccccc} 00 & 01 & 02 & 03 & 04 & 05 & 06 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 20 & 21 & 22 & 23 & 24 & 25 & 26 \\ 30 & 31 & 32 & 33 & 34 & 35 & 36 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 \\ 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 60 & 61 & 62 & 63 & 64 & 65 & 66 \end{array} \right\|$$

Each row of this matrix is transformed into itself by G_∞, each column into itself by G_0. Further the seven hyperplanes of each heptahedron are linearly independent. Let us prove this for G_∞. If the seven hyperplanes X_i are not independent, there is a relation among them involving $7-k$ of these X's such that these $7-k$ X's do not satisfy further relations. Then the $7-k$ X's in this relation are all fixed under G_∞ and meet in an S_k which is also fixed under G_∞. Hence G_∞ permutes the S_5's on the fixed S_k in such a way that $7-k$ of the S_5's are fixed. Therefore by projection from S_k upon an S_{5-k}, we should have in S_{5-k} $7-k$ fixed spaces no. $6-k$ of which were related. But such a collineation is the identity in S_{5-k}. Hence every S_5 on S_k is fixed, contrary to the fact that G_∞ has only a finite number 7 of fixed spaces.

3. **A canonical form of the $G_{2\cdot 7^2}$.** Let then the heptahedron H_∞ with linearly independent faces be chosen as a reference figure and denote these faces by X_i $(i = 0, 1, \ldots, 6)$.* These are determined by the rows of the matrix (6). The reference figure is completed by choosing a unit-hyperplane. This hyperplane will be chosen as the one containing the singular points of the first column of the matrix (6).

In terms of the coördinates thus defined the generators of the $G_{2\cdot 7^2}$ of collineations of the E^7 into itself have the form

$$(7)\qquad \begin{array}{rcll} S_{01} : X_i' & = & X_{i-1} & (X_{7+i} \equiv X_i) \\ S_{10} : X_i' & = & \epsilon^i X_i & (i = 0, 1, \ldots, 6) \\ V : X_i' & = & X_{-i} & (X_{-i} \equiv X_{7-i}) \end{array}$$

where ϵ is a seventh root of unity. The formulas (7) constitute a first canonical form of $G_{2\cdot 7^2}$.

4. **The family of E^7's.** The curve E^7 depends upon the ratio $\omega = \omega_1/\omega_2$. For each value of ω, there is an E^7, hence there is a family F of E^7's. But the

---

* X_i is written instead of $X_i(u)$. The X_i can be represented as the products of sigma functions, i. e.,

$$X_i(u) = \alpha_i \prod_{j=0}^{j=6} \sigma(u - \omega_{ij})$$

where the α_i are constants which insure the double periodicity of the ratios X_i. Compare K. F., vol. 2, p. 238.

group $G_{2\cdot 7^2}$ of collineations is the same for each member of the family F since its coefficients are numbers independent of the ratio ω. For each curve of F the set of 8 heptahedra is the same, since the heptahedra are determined by their common $G_{2\cdot 7^2}$.

All collineations which leave each member of F unaltered have been determined. If there are further collineations which interchange the members of F, they must arise from integer period transformations of determinant $+1$. Consider then the transformation

$$\begin{aligned} \omega_1' &= \alpha\omega_1 + \beta\omega_2 \\ \omega_2' &= \gamma\omega_1 + \delta\omega_2 \end{aligned} \qquad \alpha\delta - \beta\gamma = 1 \tag{8}$$

where $\alpha, \beta, \gamma, \delta$ are integers. The curve as first expressed in terms of $p(u)$ and its derivatives is unaltered by (8). In the new reference system the curve and each ω_{ij} are unaltered if (8) is congruent to identity (mod 7). On the other hand if (8) is not congruent to identity (mod 7), the ω_{ij} are permuted and we may look upon this operation either as merely a change in the coördinate system in which the curve is fixed or as a collineation in which the reference system is fixed and the E^7 passes into a new curve which belongs to F. Therefore all transformations (8) which are congruent to identity (mod 7) give rise to the identical collineation. These transformations constitute a subgroup of (8) of index $2\cdot 168$. All elements of (8) in a coset of this subgroup give rise to a collineation which permutes the curves of F except the element $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}$ which is the element V. Hence there are $2\cdot 168/2$ or 168 collineations which interchange the members of F.* These collineations may be represented by the elements of (8) reduced modulo 7, that is

$$\begin{aligned} \omega_1' &= \alpha\omega_1 + \beta\omega_2 \\ \omega_2' &= \gamma\omega_1 + \delta\omega_2 \end{aligned} \qquad \alpha\delta - \beta\gamma \equiv 1 \pmod 7. \tag{9}$$

It is well known that any transformation of the group (8) is a combination of the transformations

$$S : \omega' = \omega + 1 \qquad T : \omega' = -1/\omega, \tag{10}$$

where S is of period 7 and T is of period 2 when reduced modulo 7. Since $T^2 \equiv S^7 \equiv (ST)^3 \equiv (S^4T)^4 \equiv 1$†, these relations define a G_{168} of collineations on the reduced periods which permutes the members of the family F. Therefore we have the following theorem:

* See K. F., vol. 1, p. 398.

† It should be noted that in homogeneous form, T is of period 4, (S^4T) is of period 8. Hence V and T^2 are the same. Compare Dickson, *Linear Groups*, p. 303.

THEOREM I. *The family F of elliptic E^7's, each member of which is unaltered by $G_{2\cdot 7^3}$, is unaltered as a whole by a collineation group $G_{2\cdot 7^3\cdot 168}$ for which $G_{2\cdot 7^3}$ is an invariant subgroup. Under the group of F each curve belongs to a conjugate set of* 168 *curves.**

The collineation T permutes the H_i ($i = \infty, 0, 1, \ldots, 6$) as follows: ($\infty$ 0), (16), (25), (34), where the subscripts only are used. The collineation S permutes the H_i ($i = 0, 1, \ldots, 6$) cyclically and leaves H_∞ invariant. Under the group (8) the H_i are permuted like the 8 points $\infty, 0, 1, \ldots, 6$ in a finite geometry modulo 7, there being 8 points on a line.

The equations of the collineations S and T in terms of X_i are†

$$(11)\qquad \begin{aligned} S &: X_i' = \epsilon^{-i^2/2} X_i \\ T &: X_i' = c\sum_{\beta=0}^{6} \epsilon^{i\beta} X_\beta \qquad (i = 0, 1, \ldots, 6). \end{aligned}$$

5. **The fixed spaces.** In $G_{2\cdot 7^3}$, the 7^2 involutions V_{ij} ($i, j = 0, 1, \ldots, 6$) have the form $u' = -u + \omega_{ij}$. The fixed points of these involutions are $u \equiv \omega_{ij}/2 + P/2$ where $P/2$ can evidently have the values 0, $\omega_1/2$, $\omega_2/2$, and $(\omega_1 + \omega_2)/2$. We consider the simplest set, i. e., the set for which $i = j = 0$.

Due to the involutory character of V, there are two skew spaces of fixed points in S_6, an S_2 and an S_3. If the coördinates of these fixed spaces be denoted by y_i and z_j ($i = 0, 1, 2, 4$; $j = 1, 2, 4$) respectively, the equations of the transformation from the coördinates X_i to those of y and z are

$$(12)\qquad \begin{aligned} X_0 &= y_0, \\ X_1 + X_6 &= 2y_1, & X_1 - X_6 &= 2z_1, \\ X_2 + X_5 &= 2y_2, & X_2 - X_5 &= 2z_2, \\ X_4 + X_3 &= 2y_4, & X_4 - X_3 &= 2z_4. \end{aligned}$$

In terms of y and z, V now has the form

$$(13)\qquad y_i' = y_i, \qquad z_j' = -z_j \qquad (i = 0, 1, 2, 4;\ j = 1, 2, 4).$$

In (12), $y_i = 0$ determine the S_2 of fixed points and $z_j = 0$ determine the S_3 of fixed points. The fixed S_5's are either on S_2 with equations $\alpha_0 y_0 + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_4 y_4 = 0$ or on S_3 with equations $\beta_1 z_1 + \beta_2 z_2 + \beta_4 z_4 = 0$. The α's may be determined by putting the S_5 on u_1, u_2, u_3, three arbitrary points on E^7, so that necessarily this S_5 cuts E^7 in the points $-u_1$, $-u_2$, $-u_3$. Therefore the S_5 contains the point $u = 0$, but no proper half period point. Hence all the fixed S_5's on the S_2 and therefore S_2 itself, contain the point $u = 0$ but no proper

* See K. F., vol. 1, p. 398.

† Compare K. F., vol. 2, p. 292. The formula for S is compatible with Klein's for n a prime number. As we deal with collineations in homogeneous forms we do not need to keep c of the K. F. formula; it is therefore dropped in the remainder of the work.

half period point. Therefore S_3 contains the proper half period points since they are also fixed points.

The family F of E^7's projected from the fixed S_2 upon the fixed S_3 becomes a family F_1 of rational cubics doubly covered, since the pairs $(\pm u)$ corresponding under V each project into the same point. In a similar manner, by projection from S_3 upon S_2, F becomes a family F_2 of conics doubly covered.

It is my purpose to discuss the families F_1, F_2, for which the curves in each family will vary with ω whereas the points on a particular curve will vary with the pairs $(\pm u)$ on the original E^7. The $G_{2\cdot 7^2\cdot 168}$ has now reduced to a G_{168} in S_2 and S_3 which leaves F_1 and F_2 invariant. This G_{168} is generated by S and T whose equations are easily found to be

$$(14)\qquad \begin{array}{lll} S & : & \begin{array}{l} y_i' = \epsilon^{-i^2/2} y_i \\ z_j' = \epsilon^{-j/2} z_j \end{array} \qquad (i = 0, 1, 2, 4;\ j = 1, 2, 4) \\ T & : & \begin{array}{l} y_i' = y_0 + \sum_j (\epsilon^{ij} + \epsilon^{-ij})\, y_j \\ z_k' = \sum_l (\epsilon^{lk} - \epsilon^{-lk})\, z_l \end{array} \qquad \begin{array}{l} (i = 0, 1, 2, 4) \\ (j, k, l = 1, 2, 4). \end{array} \end{array}$$

Formulas (12), (13) and (14) constitute a second canonical system of coördinates for E^7.

II. The quadrics on E^7

1. **The pencil of quadrics on E^7.** Hermite has shown that the number of linearly independent quadrics on E^7 is fourteen. These fourteen quadrics cut out the E^7 completely with no extraneous intersection.* In the second system of coördinates a general quadric has the form

$$(15)\qquad q_0 = \sum_{i,k=0}^{6} \alpha_{ik} X_i X_k = 0,$$

where α_{ik} are constants. Let us suppose that the α_{ik} are so determined that the quadric contains the curve E^7. Under the collineation S_{10}, E^7 is transformed into itself. Hence the quadric (15) is transformed into a quadric on E^7. The transforms of q_0 under S_{10} are of the form

$$(16)\qquad q_j = \sum_{i,k=0}^{6} \alpha_{ik}\, \epsilon^{j(i+k)} X_i X_k = 0 \qquad (j = 0, 1, \ldots, 6).$$

Since each q_j is on E^7, a linear combination of them will be on the curve. Multiplying each q_j by unity and adding we obtain a *particular* quadric Q_0 on E^7 characterized by the fact that it consists only of those terms for which $i + k \equiv 0 \pmod 7$. Using the multipliers 1, ϵ^5, ϵ^3, ϵ^1, ϵ^6, ϵ^4, ϵ^2, respectively, we obtain a

* Compare K. F., vol. 2, p. 245.

second *particular* quadric Q_1 on E^7 characterized by the fact that it consists only of those terms for which $i + k \equiv 1 \pmod 7$. Proceeding in this way we obtain 7 *particular* quadrics on E^7. They are

$$(17)\quad Q_i = \alpha_{i0}X_i^2 + 2\alpha_{i1}X_{i+1}X_{i-1} + 2\alpha_{i2}X_{i+2}X_{i-2} + 2\alpha_{i4}X_{i+4}X_{i4} = 0 \quad (i = 0, 1, \ldots, 6).$$

Any quadric on the curve E^7 is a linear combination of the Q's, since the seven Q's contain as yet 28 arbitrary coefficients. But since each Q_i is sent into Q_{i+1} by S_{01}, these 28 coefficients reduce to four, i. e., α_0, α_1, α_2, α_4. From these seven Q's, we know that we must be able to get the 14 linearly independent quadrics on the E^7. The α's therefore must contain a parameter linearly and there will be one quadric of the type Q_i for which a particular α will vanish.* At most, then, a pencil can arise from the four terms of each Q_i. Any one of these seven pencils is defined by the fact that it admits one of the seven dihedral $G_{2\cdot 7}$'s whose cyclic subgroup is S_{10}. For example Q_0 admits the dihedral $(S_{10}V)$.

Since the α's contain a parameter linearly, they may be interpreted as the coördinates of a point on a line in an S_3. By choosing two members from the pencil of quadrics, the line is determined. We shall determine the α's later as functions of ω and the parameter just mentioned.

2. **The group on the quadrics.** Under $G_{2\cdot 7^2}$ each member of the family F of E^7's is transformed into itself and the quadrics on each curve are transformed into quadrics on that curve, so that a group of collineations is induced upon the Q_i as variables. Moreover since S and T interchange the members of the 168 sets of conjugate curves, they will send the quadrics on a given curve into a linear combination of the quadrics on the transformed curve. If we indicate the group $G_{2\cdot 7^2\cdot 168}$ on the X_i's in (7) and (11) by $G(\epsilon)$, then the induced group on the quadrics Q_i is $G(\epsilon^2)$.

In order to express all the quadrics (17) by one equation, consider the general quadric obtained by taking a linear combination of them. Such a quadric has the form.

$$(18)\quad \sum_{i=0}^{6} L_i Q_i = 0,$$

where the L_i are arbitrary constants. On a given curve of F determined by a proper set of values of α_i $(i = 0, 1, 2, 4)$, the bilinear form (18) is an identity in L and u. If we require that this bilinear form be an invariant under $G(\epsilon^2)$, there will be a certain group induced upon the L_i as variables. This group on the variables L_i is $G(\epsilon^{-2})$.

* Compare K. F., vol. 2, p. 268. Klein obtained the quadrics on the elliptic curves from the three-term sigma relation.

3. **A Kleinian form.** Since the properties of the groups on the L_i and Q_i are the same as those on the X_i, we isolate one of the involutions in the L_i, Q_i groups, i. e., that one induced by V which was isolated in the X_i group. We introduce the variables v and u, ζ and ϑ with Q_i and L_i, respectively, as y and z were introduced with the X_i. The equations of the transformations from Q_i and L_i to v, u, ζ and ϑ can be written down as were those for y and z. After this change of variables, (18) has the form

$$
\begin{aligned}
F' = \quad & \zeta_0[\alpha_0 y_0^2 + 2\alpha_1 y_1^2 + 2\alpha_2 y_2^2 + 2\alpha_4 y_4^2 - 2\alpha_1 z_1^2 - 2\alpha_2 z_2^2 - 2\alpha_4 z_4^2] \\
& + 2\,\zeta_1[\alpha_0 y_1^2 + 2\alpha_1 y_0 y_2 + 2\alpha_2 y_1 y_4 + 2\alpha_4 y_2 y_4 + \alpha_0 z_1^2 + 2\alpha_2 z_1 z_4 - 2\alpha_4 z_2 z_4] \\
& + 2\,\zeta_2[\alpha_0 y_2^2 + 2\alpha_1 y_1 y_4 + 2\alpha_2 y_0 y_4 + 2\,\alpha_4 y_1 y_2 - 2\,\alpha_1 z_1 z_4 + \alpha_0 z_2^2 + 2\alpha_4 z_1 z_2] \\
& + 2\,\zeta_4[\alpha_0 y_4^2 + 2\alpha_1 y_2 y_4 + 2\alpha_2 y_1 y_2 + 2\alpha_4 y_0 y_1 + 2\alpha_1 z_2 z_4 - 2\alpha_2 z_1 z_2 + \alpha_0 z_4^2] \\
& + 4\,\vartheta_1[\alpha_0 y_1 z_1 + \alpha_1 y_0 z_2 - \alpha_2(y_1 z_4 + y_4 z_1) + \alpha_4(y_2 z_4 - y_4 z_2)] \\
& + 4\,\vartheta_2[\alpha_0 y_2 z_2 + \alpha_1(y_4 z_1 - y_1 z_4) + \alpha_2 y_0 z_4 - \alpha_4(y_1 z_2 + y_2 z_1)] \\
& + 4\,\vartheta_4\,[\alpha_0 y_4 z_4 - \alpha_1(y_2 z_4 + y_4 z_2) + \alpha_2(y_1 z_2 - y_2 z_1) + \alpha_4 y_0 z_1] = 0.
\end{aligned} \tag{19}
$$

On E^7 the above form is an identity in ζ, ϑ and can be separated into seven parts. However we shall have occasion to separate it into two parts, P_1 and P_2, such that the part P_1 contains the coefficients ζ and the part P_2 the coefficients ϑ. The part P_1 is partly symmetrical and partly alternating in the coefficients α and ζ, hence the ζ's can be interpreted as the coördinates of a point on a line in an S_3 and are therefore cogredient to the α's. Hence we may conclude this section with the theorem

THEOREM II. *F' is a Kleinian form which remains invariant under the simultaneous transformation by the isomorphic groups $M(\epsilon)$ of (14) on the variables y and z; $M(\epsilon^{-2})$ on the variables ζ and α and ϑ. The form F' determines the curve E^7 uniquely when the modular functions α are properly given, i. e., subject to the relation which connects their ratios.*

III. THE INTERPRETATION OF THE FORM F'

1. Its fundamental elliptic modular functions. Each curve of the family F has on it the point whose parameter is $u = 0$. As $\omega = \omega_1/\omega_2$ varies this zero point generates a locus. It has already been pointed out that the zero point is in the space S_2 of fixed points, i. e., when $u = 0$ all the y's vanish. Let $z_i = t_i$ $(i = 1, 2, 4)$ for $u = 0$; then F' in (19) reduces to

$$
\begin{aligned}
& \zeta_0[0 - 2\,\alpha_1 t_1^2 - 2\,\alpha_2 t_2^2 - 2\,\alpha_4 t_4^2] \\
& \quad + 2\,\zeta_1[\alpha_0 t_1^2 - 0 + 2\,\alpha_2 t_1 t_4 - 2\,\alpha_4 t_2 t_4] \\
& \quad + 2\,\zeta_2[\alpha_0 t_2^2 - 2\,\alpha_1 t_1 t_4 - 0 + 2\,\alpha_4 t_1 t_2] \\
& \quad + 2\,\zeta_4[\alpha_0 t_4^2 + 2\,\alpha_1 t_2 t_4 - 2\,\alpha_2 t_1 t_2 - 0] \equiv 0.
\end{aligned} \tag{20}
$$

Since (20) is an identity in the ζ_i, their coefficients must vanish. These coefficients are linear in the α's, all of which do not vanish simultaneously, therefore the determinant of the α's must vanish. After removing numerical factors, we find a skew-symmetric determinant of even order. This determinant is a perfect square.* It furnishes in variables t_i Klein's quartic, which is denoted as follows:

$$K = t_1^3 t_2 + t_2^3 t_4 + t_4^3 t_1 = 0. \tag{21}$$

K is the equation of the locus of the zero point of the family of E^7's and admits a group G_{168} of collineations into itself, cogredient to the group in z in (14). The ratios $t_1 : t_2 : t_4$ are the *fundamental elliptic modular functions of the form* F'. The expressions for these ratios as uniform functions of the modulus ω may be obtained by setting $u = 0$ in the expressions for the z's in terms of u, ω_1, ω_2, as indicated.

Since the curve E^7 varies with ω, and since each E^7 possesses a zero point, i. e., a point t which is on K, it is clear that the variation of E^7 with ω may be imaged by the variation of t on K. We shall express other elliptic modular functions associated with the family of E^7's in terms of the t_i.

2. **The null-system.** The form in (20) is a null-system, since it can be written in the form

$$(\alpha_0\zeta_1)t_1^2 + (\alpha_0\zeta_2)t_2^2 + (\alpha_0\zeta_4)t_4^2 + 2(\alpha_4\zeta_2)t_1t_2 + 2(\alpha_1\zeta_4)t_2t_4 + 2(\alpha_2\zeta_1)t_4t_1 = 0, \tag{22}$$

where $(\alpha_i\zeta_k) = \alpha_i\zeta_k - \alpha_k\zeta_i$. Since (20) vanishes independently of the ζ's it represents a *singular null-system.*† Hence (22) is the equation of a line whose coördinates may be taken as

$$\begin{aligned} (\alpha_0\zeta_1) &= +\,2\,t_1t_2, & (\alpha_4\zeta_2) &= t_1^2, \\ (\alpha_0\zeta_2) &= +\,2\,t_2t_4, & (\alpha_1\zeta_4) &= t_2^2, \\ (\alpha_0\zeta_4) &= +\,2\,t_4t_1, & (\alpha_2\zeta_1) &= t_4^2, \end{aligned} \tag{23}$$

where α is clearly a point on a line. Since the coördinates of the line of the α's are functions of t, we shall call it the *modular line* and denote it by L_m. The intersection of the coördinate planes of the reference tetrahedron in the space of the α's, an $S_3^{(\alpha)}$, with L_m furnishes four convenient sets of values of the α's, which substituted in F' give rise to the 28 quadrics on E^7, of which only 14 are linearly independent, since any two sets of the α's are linear combinations of the remaining two sets. These sets of values are

* Burnside and Panton, *Theory of Equations*, vol. 2, p. 46.

† See Veblen and Young, *Projective Geometry*, vol. 1, p. 324.

$$(24)\qquad \alpha_0 : \alpha_1 : \alpha_2 : \alpha_4 = \begin{vmatrix} 0 & -2l_1l_2 & -2l_2l_4 & -2l_1l_4 \\ 2l_1l_2 & 0 & +l_4^2 & -l_2^2 \\ 2l_2l_4 & -l_4^2 & 0 & l_1^2 \\ 2l_4l_1 & l_2^2 & -l_1^2 & 0 \end{vmatrix}$$

The sets (24) suggest that we make a transformation on the α's in F'. Let ξ be a plane such that it intersects L_m in the point α. From (24) we find this transformation to be

$$(25)\qquad \begin{aligned} \alpha_0 &= 0 + 2l_1l_2\xi_1 + 2l_2l_4\xi_2 + 2l_4l_1\xi_4, \\ \alpha_1 &= -2l_1l_2\xi_0 + 0 - l_4^2\xi_2 + l_2^2\xi_4, \\ \alpha_2 &= -2l_2l_4\xi_0 + l_4^2\xi_1 + 0 - l_1^2\xi_4, \\ \alpha_4 &= -2l_4l_1\xi_0 - l_2^2\xi_1 + l_1^2\xi_2 + 0. \end{aligned}$$

If F' is transformed by (25), it will take the form

$$(26)\qquad \sum \xi_i\,\zeta_j\,\phi_{ij} + \sum \xi_i\,\vartheta_l\,\phi'_{il} = 0 \qquad (i, j = 0, 1, 2, 4\ ;\ l = 1, 2, 4).$$

The 28 quadrics on the curve E^7, of which naturally only 14 are linearly independent, are found by equating to zero the coefficients of the terms $\xi_i\,\zeta_j$ and $\xi_i\,\vartheta_l$ respectively, i. e., the ϕ_{ij} and ϕ'_{il}. We shall have occasion to use all of these quadrics, but will refer to them briefly in the above notation.

3. **The rational curves in S_2 and S_3.** We have seen that α and ζ are cogredient variables and that P_1 is partly alternating and partly symmetrical in α and ζ. We now rewrite P_1 so as to exhibit this property. It has the form

$$(27)\qquad \begin{aligned} &\alpha_0\zeta_0y_0^2 + 4\sum\alpha_1\zeta_1y_0y_2 + 2\sum(\alpha_0\zeta_1 + \alpha_1\zeta_0)y_1^2 + 4\sum(\alpha_2\zeta_4 + \alpha_4\zeta_2)y_1y_2 \\ &\qquad + 2\sum(\alpha_0\zeta_1)z_1^2 + 4\sum(\alpha_4\zeta_2)\,z_1z_2 = 0, \end{aligned}$$

where Σ, unless otherwise denoted, refers to the cyclic advance of the subscripts 1, 2, 4. This form furnishes the means by which the projections of the family F of E^7's upon the fixed spaces S_2 and S_3 are found. The second part P_2, bilinear in y and z, does not enter in these projections, since it vanishes when either space is considered separately.

Since ζ is perfectly arbitrary, consider it on the modular line L_m. Now interchange α and ζ in (27). The new form is similar to the old except that the sign of each term in z is changed. Denote the transformed P_1 by P_1'. Since P_1 in (27) is a quadric on E^7 and since we consider ζ on L_m, P_1' is also a quadric on E^7. Whence their sum $P_1 + P_1'$ and their difference $P_1 - P_1'$ are quadrics on E^7. Consider the former;

$$(28)\qquad \alpha_0\zeta_0y_0^2 + 4\sum\alpha_1\zeta_1y_0y_2 + 2\sum(\alpha_0\zeta_1 + \alpha_1\zeta_0)y_1^2 + 4\sum(\alpha_2\zeta_4 + \alpha_4\zeta_2)y_1y_2 = 0.$$

The equation (28) for arbitrary α and ζ on L_m furnishes a system of quadrics in S_3 which intersect in a cubic curve. From the symmetry of α and ζ in (28), we lose no generality by setting $\alpha_i = \zeta_i$. We then have

$$(29) \qquad \alpha_0^2 y_0^2 + 4\sum \alpha_1^2 y_0 y_2 + 4\sum \alpha_0\alpha_1 y_1^2 + 8\sum \alpha_2\alpha_4 y_1 y_2 = 0.$$

Since α is linear in a parameter λ on L_m, (29) furnishes a system of quadrics quadratic in λ. The coefficients of this quadratic system of quadrics are functions of t, so that as t varies on K, we get a family F_1 of cubic curves C^3 in S_3. Hence we may state the following theorem:

(30) THEOREM III. *The projection C^3 of the curve E^7 upon S_3 is the base curve of the quadratic system of quadrics* (29).

Consider now the difference $P_1 - P_1'$. This is a conic in S_2. It has the form

$$(31) \qquad \sum(\alpha_0\zeta_1)z_1^2 + 2\sum(\alpha_1\zeta_2)z_1z_2 = 0,$$

which from (23) may be written as follows:

$$(32) \qquad \sum t_1t_2z_1^2 + \sum t_1^2z_1z_2 = 0.$$

This shows that the system of conics varies with t on K. It is the polar conic of K as to z. Hence the theorem:

(33) THEOREM IV. *The projection C^2 of the family F of E^7's upon S_2 is the system of polar conics of Klein's quartic K.*

4. **The net of quadrics in $S_3(\alpha)$.** The quadric in (29) will be the square of a plane when the rank of its discriminant is 1. Its discriminant is of rank 1 if only the three relations

$$(34) \qquad \alpha_0\alpha_2 - \alpha_3^2 = 0, \qquad \alpha_0\alpha_1 - \alpha_2^2 = 0, \qquad \alpha_0\alpha_1 - \alpha_2^2 = 0,$$

are satisfied.

Consider now the net of quadrics

$$(35) \qquad t_1(\alpha_0\alpha_2 - \alpha_4^2) + t_2(\alpha_0\alpha_4 - \alpha_1^2) + t_4(\alpha_0\alpha_1 - \alpha_2^2) = 0.$$

From the transformations S and T in (14) we conclude that (35) is a Kleinian form.

The discriminant of the net (35) is K. Hence so long as t is on K, the quadric (35) has a double point. If we border the discriminant (35) with variables ξ and expand, we find the equation of this double point to be

$$(36) \quad (\alpha\xi)^2 = \xi_0^2 4t_1t_2t_4 + \sum(-t_1^3 - t_2^2t_4)\,\xi_1^2 + \sum(2_1tt_1^2)(2\xi_0\xi_1) + \sum(2\,t_1^2t_4)(2\xi_1\xi_2) = 0.$$

Therefore the coördinates of the double-point are

$$
\begin{aligned}
\rho\,\alpha_0 &= \sqrt{4t_1t_2t_4} &&: 2t_1t_2t_4 : && t_1t_4^2 && : && t_1^2t_2 && : && t_2^2t_4 \\
\rho\,\alpha_1 &= \sqrt{-t_1^3-t_2^2t_4} && : && t_1t_4^2 && : -t_1^3-t_2^2t_4 && : && t_1^2t_4 && : && t_4^2t_2 \\
\rho\,\alpha_2 &= \sqrt{-t_2^3-t_4^2t_1} && : && t_2t_1^2 && : && t_1^2t_4 && : -t_2^3-t_4^2t_1 && : && t_2^2t_1 \\
\rho\,\alpha_4 &= \sqrt{-t_4^3-t_1^2t_2} && : && t_4t_2^2 && : && t_4^2t_2 && : && t_2^2t_1 && : -t_4^3- && t_1^2t_2
\end{aligned}
\tag{37}
$$

where ρ is 1, α_0, α_1, α_2, α_4, respectively. That is to say, we can express the entire system (37) rationally and without extraneous factors by giving the ten quadratic combinations of the α's. These combinations are the coefficients of the terms $\xi_i\xi_k$ in (36).

The order of the linear modular group in the space of the y's and α's is double the order of the group* in the space of the z's, that is, the group is a $G_{2\cdot 168}$, due to the fact that the identical collineation appears in the form $y_i' = \pm y_i$. Hence the coördinates of a modular-point or plane in $S_3^{(\alpha)}$ and likewise in S_3 cannot be expressed rationally in terms of the t_i without an extraneous factor. The coördinates may however be expressed irrationally in terms of t as above, and it is to be noted that their ratios are uniform functions of ω.

A number of such modular root functions are suggested by the geometry of the system of cubic curves C^3 in S_3. Thus the locus of the zero point on the curves C^3, the locus of the plane of the half period points, the locus of the point where the tangent at the zero point meets the half period plane, as well as the transforms of these points and planes in the null-system of C^3, give rise to functions of this type. Some of these are determined later.

The locus of the double point (36) as t varies on K is a well known space curve J of order 6 in $S_3^{(\alpha)}$,† whose points are in a one-to-one correspondence with the points of K. If we border the discriminant of (35) with ξ and η, which are to be thought of as parameters, we have ∞^6 curves of the third order in t which intersect K in 12 points which correspond to the 12 meets of the planes ξ, η with J. Hence when $\xi = \eta$ the cubic in t will be a contact cubic of K. Thus the system (36) for variable ξ is a system of contact curves of the third order associated with J.‡

5. **The modular line and spread.** If a point y be taken on J, a quadric of the net (35) has a node at y and the polar plane of this point as to this quadric vanishes, while the polar planes of the other two quadrics meet in a line. Take the coördinates of the point y on J as those in the second column of (37). The three

* Compare K. F., vol. 2, p. 313.

† Compare Snyder and Sisam, *Analytic Geometry of Space*, p. 168.

‡ See K. F., vol. 1, p. 716.

polar planes of this point as to the quadrics in the net (35) are in a pencil, and have the form

$$\begin{aligned} &\alpha_0 l_1^2 - 0 + 2\,\alpha_2 l_1 l_4 - 2\,\alpha_4 l_2 l_4 = 0, \\ &\alpha_0 l_2^2 - 2\,\alpha_1 l_1 l_4 - 0 + 2\,\alpha_4 l_1 l_2 = 0, \\ &\alpha_0 l_4^2 + 2\,\alpha_1 l_2 l_4 - 2\,\alpha_2 l_1 l_2 - 0 = 0. \end{aligned} \tag{38}$$

The axis of the pencil of planes (38) *is the modular line* L_m. Every point on L_m is in a one-to-one correspondence with the point y on J. Since the coördinates of L_m and of the point y on J are functions of t, the variation of y and of L_m also may be imaged by the variation of t on K. Hence as y generates J, L_m generates a ruled surface of order 8. That M is of order 8 may be shown as follows. The condition that a line l meet L_m is a linear condition on their coördinates, or a conic in t. This conic in t meets K in 8 points to each of which there corresponds a meet of l and M, whence M is of order 8.

Let us now consider the general quadric Q in the net (35), and put on it the condition that it have a node. The four partial derivatives $\partial Q/\partial \alpha_i$ must then vanish simultaneously. These are

$$\begin{aligned} 0 + \alpha_1 l_4 + \alpha_2 l_1 + \alpha_4 l_2 &= 0, \\ \alpha_0 l_4 - 2\alpha_1 l_2 &= 0, \\ \alpha_0 l_1 - 2\alpha_2 l_4 &= 0, \\ \alpha_0 l_2 - 2\alpha_4 l_1 &= 0. \end{aligned} \tag{39}$$

The discriminant of these equations is K. If we eliminate t from the equations (39), we find four cubic surfaces on each of which is J. Hence their common intersection is J. The equations of these are obtained from the vanishing of the third order determinants in the matrix of the equations (39). They are

$$\begin{aligned} S_1 &= \alpha_0^3 - 8\,\alpha_1\alpha_2\alpha_4 = 0, \\ S_2 &= \alpha_0^2\alpha_4 + 2\,\alpha_0\alpha_2^2 + 4\,\alpha_1\alpha_2^2 = 0, \\ S_3 &= \alpha_0^2\alpha_1 + 2\,\alpha_0\alpha_4^2 + 4\,\alpha_2\alpha_4^2 = 0, \\ S_4 &= \alpha_0^2\alpha_2 + 2\,\alpha_0\alpha_1^2 + 4\,\alpha_4\alpha_1^2 = 0. \end{aligned} \tag{40}$$

The modular spread M multiplied by α_0 is the following combination of S_i in (40):

$$S_1^3 - 8\,S_2S_3S_4 = \alpha_0 M = 0. \tag{41}$$

From this result it is evident that J *is a triple curve on* M. Further, it can be shown that *through every point of* J *there pass three trisecants of* J *and that* L_m *itself is a trisecant of* J.*

This section can be partially summarized in the following theorem:

* The equation of M and the facts concerning J are easily obtained from a Cremona transformation of the third order.

THEOREM V. *Through every point* α $(= \alpha_0, \alpha_1, \alpha_2, \alpha_4)$ *on the octavic ruled surface* M *there passes a line* L_m *and the pencil of points* α *on* L_m *set in the form* F' *determines the quadrics on the curve* E^7. *As the line* L_m *varies on* M, *the* E^7 *varies in the family* F. *The line* L_m *(itself a trisecant of* J*) meets the triple curve* J *on* M *in three points which correspond to the three trisecants of* J *that meet in a point* t *of* J. *Thus the points* t *of* J *are in a one-to-one correspondence with the curves of the family of* E^7*'s.*

This completes the determination of the coefficients α of the quadrics F' which define the curve E^7.

IV. THE LOCI IN S_3

1. **The net of quadrics in** S_3. In (35) a net of quadrics in $S_3^{(\alpha)}$ was considered. The modular line L_m and the modular spread were associated with this net. Consider now a similar net of quadrics in plane coördinates U in S_3, and let us find the condition that this net have a double plane. From the contragredient transformations S and T on the y's in (14)*, we conclude that the following net is a Kleinian form:

$$t_1(2\,U_0U_1 - U_2^2) + t_2(2\,U_0U_2 - U_4^2) + t_4(2\,U_0U_4 - U_1^2) = 0. \tag{42}$$

The discriminant of this net is K. The bordered form of the discriminant is the square of a plane in point coördinates, i. e.,

$$t_1t_2t_4y_0^2 + \sum(-t_2^3 - t_1t_4^2)y_1^2 + 2\sum t_1^2t_2y_0y_1 + 2\sum t_1t_2^2y_1y_2 = 0. \tag{43}$$

So long as t is on K, the coördinates of the double plane (43) are

$$\begin{aligned}
\rho U_0 &= \sqrt{t_1t_2t_4} &&: t_1t_2t_4 : t_1^2t_2 : t_2^2t_4 : t_4^2t_1,\\
\rho U_1 &= \sqrt{-t_2^3 - t_1t_4^2} &&: t_1^2t_2 : -t_2^3 - t_1t_4^2 : t_1t_2^2 : t_1^2t_4,\\
\rho U_2 &= \sqrt{-t_4^3 - t_2t_1^2} &&: t_2^2t_4 : t_1t_2^2 : -t_4^3 - t_2t_1^2 : t_2t_4^2,\\
\rho U_4 &= \sqrt{-t_1^3 - t_4t_2^2} &&: t_4^2t_1 : t_1^2t_4 : t_2t_4^2 : -t_1^3 - t_4t_2^2,
\end{aligned} \tag{44}$$

where ρ is 1, U_0, U_1, U_2, U_4, respectively. As in (37), we may express the entire system in (44) by taking the 10 quadratic combinations of the U's from (43). The remarks following (37) apply here. The plane coördinates U_i taken from the second column of (44) are the modular systems A_γ developed by Klein.†

With the net (42) there will be a modular line L_m', four cubic surfaces S_i', a modular surface M' and a sextic J'. The coördinates of L_m' can be developed

* See K. F., vol. 1, p. 719.

† See K. F., vol. 1, p. 719.

in a manner similar to that used in finding those of L_m as the axis of the pencil of planes (38). They are

$$(45)\qquad \begin{aligned} (U_0U_1') &= t_1t_2, & (U_2U_1') &= t_1^2, \\ (U_0U_2') &= t_2t_4, & (U_4U_2') &= t_2^2, \\ (U_0U_4') &= t_4t_1, & (U_1U_4') &= t_4^2. \end{aligned}$$

To every position of the plane $(Uy) = 0$ in (43) we have a line L_m' whose coördinates are given in (45). Since the coefficients of the plane $(Uy) = 0$ and L_m' are functions of t, the variation of the plane (Uy) and L_m' also may be imaged as the variation of t on K. It should be noted that the space of the α's is different from the space of the y's. Hence the modular lines L_m, L_m'; the curves J, J'; the spreads M, M'; and the cubic surfaces S_i, S_i' are all distinct.

2. **Tne plane of the half period points.** For the three half period points, the z's all vanish. If in the 14 linearly independent quadrics on E^7 we set the z's all zero, we then obtain 8 quadrics in y (since 6 of the 14 quadrics are bilinear in y and z and vanish for z set equal zero). These 8 quadrics must pass through the half period points. If we call the plane of these points (Uy), then we should be able to obtain from these 8 quadrics the four combinations $y_i(Uy)$ $(i = 0, 1, 2, 4)$. The combinations furnishing these types of quadrics come from the systems

$$(46)^*\qquad \begin{aligned} t_1t_4^3\phi_{12} + t_1t_2^2t_4\phi_{14} - t_1^2t_2^2\phi_{24} - t_2^2t_4^2\phi_{21} + t_2t_4^3\phi_{40} &= 0, \\ t_1\phi_{41} + t_2\phi_{01} - t_4\phi_{04} &= 0, \\ t_2\phi_{12} + t_4\phi_{02} - t_1\phi_{01} &= 0, \\ t_4\phi_{24} + t_1\phi_{04} - t_2\phi_{02} &= 0. \end{aligned}$$

The common factor (Uy) obtained from these equations (46) when the z's are zero is precisely the plane

$$(47)\qquad (Uy) = t_1t_2t_4ty_0 + t_1^2t_2y_1 + t_2^2t_4y_2 + t_4^2t_1y_4 = 0,$$

whose square appeared in (43). Hence *the coördinates of the half period plane are the modular functions set forth in* (44).

Since the half period plane is of the form $\Sigma\ \alpha_iy_i = 0$ $(i = 0, 1, 2, 4)$, and since it may be considered as an S_5 in S_6, it contains the point $u = 0$ and three pairs of points $(\pm u)$ on E^7, since the three pairs are sufficient to determine the α's. It is therefore a fixed S_5 on the fixed S_2. Since the pairs $(\pm u)$ are the half period points, they are coincident points in S_6, hence *the half period plane* (47) *considered as an* S_5 *is a tritangent hyperplane of* E^7, tangent at the points $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$ and passing through the point $u = 0$.

* We draw from the entire system of quadrics ϕ_{ij}(16 in number) for convenience. These ϕ_{ij} are the coefficients of the terms $\xi_i\xi_j$ in F' after the transformation in (25).

Let us now consider the systems of quadrics in (46) with the z's different from zero. These expressed in terms of y and z are

$$
\begin{aligned}
t_4 y_0(Uy) &= +\,2t_1t_2^2z_2\ (41) + (-\,2t_1t_2t_4z_2 - 2t_2^2t_4z_1)\ (12),\\
t_4 y_1(Uy) &= -\,t_2^2t_4z_4\ (12) + (t_2^2t_4z_2 - t_1^2t_2z_1)\ (41),\\
t_1 y_2(Uy) &= -\,t_4^2t_1z_1\ (24) + (t_4^2t_1z_4 - t_2^2t_4z_2)\ (12),\\
t_2 y_4(Uy) &= -\,t_1^2t_2z_2\ (41) + (t_1^2t_2z_1 - t_4^2t_1z_4)\ (24),
\end{aligned}
\tag{48}
$$

where $(ik) = t_iz_k - t_kz_i$. Each of the above quadrics vanishes for $z_i = t_i$, that is each conic on the right in (48) intersects the polar conic C^2 in the zero point. The three remaining variable intersections of these conics and the polar conic correspond to the intersection of the plane $(\alpha_i y_i) = 0$ and the curve C^3 in S_3. Hence the system of quadrics (48) give a parametric representation of the curve C^3. To each z in (48) there is a definite point y in S_3 except at the base point of the system $z_i = t_i$. This representation can be put in a simpler form if we multiply the quadrics in (48) by t_it_k, so that each quadric on the left has the common factor $t_1t_2t_4(Uy)$, which may be dropped, leaving the parametric representation of the curve C^3 as follows:

$$
\begin{aligned}
y_0 &= \rho[+\,2t_1^2t_2^3z_2\ (41) + (-\,2\,t_1^2t_2^2t_4\,z_2 - 2t_1t_2^3t_4z_1)\ (12)],\\
y_1 &= \rho[-\,t_1t_2^3t_4z_4\ (12) + (t_1t_2^3t_4z_2 - t_1^3t_2^2z_1)\ (41)],\\
y_2 &= \rho[-\,t_2t_4^3t_1z_1\ (24) + (t_2t_4^3t_1z_4 - t_2^3t_4^2z_2)\ (12)],\\
y_4 &= \rho[-\,t_4t_1^3t_2z_2\ (41) + (t_4t_1^3t_2z_1 - t_4^3t_1^2z_4)\ (24)].
\end{aligned}
\tag{49}
$$

Hence *the doubled C^2 in S_2 is mapped upon the doubled C^3 in the fixed S_3 by means of the equations in* (49).

3. **The locus of the zero point in S_3.** In S_2 we find K as the locus of the zero point. Each curve of the system C^2 has one such point, which generates K by the variation of ω. Each curve of the system C^3 has on it the zero point. What is the equation of its locus? Since $z_i = t_i$ is the base point of the mapping system which maps C^2 upon C^3, all the y's vanish at this point, but as z approaches t the limiting position of the direction is that of the tangent to the polar conic C^2 at $z_i = t_i$. If the factors (ik) in (49) are replaced by the coördinates of the tangent to the polar conic at the point $z_i = t_i$, and if we set $z_i = t_i$ in the other factors, the y's do not vanish, and become nonic functions of t which have a common factor $t_1t_2t_4$. However, a much simpler way to get this parametric representation of the locus of the zero point in S_3 is to solve the bilinear forms $\phi'_{01} = \phi'_{02} = \phi'_{04} = 0$ for y_i, and put $z_i = t_i$ in the result, from which the factor $t_1t_2t_4$ can be removed. These equations are:

$$
\begin{aligned}
y_0 &= -\,14t_1^2t_2^2t_4^2,\\
y_1 &= t_1^4t_4^2 - 3t_1^3t_2^3 - 5t_1t_2^2t_4^3,\\
y_2 &= t_2^4t_1^2 - 3t_2^3t_4^3 - 5t_2t_4^2t_1^3,\\
y_4 &= t_4^4t_2^2 - 3t_4^3t_1^3 - 5t_4t_1^2t_2^3.
\end{aligned}
\tag{50}
$$

These equations map the locus of the zero point in S_2 upon a locus in the space of the y's. The order of this locus is 18, for a plane section $(U'y) = 0$ gives a sextic in t which intersects K in 24 points, but we find that this variable sextic and K have 6 fixed intersections at the flex points $t_i = t_k = 0$ and consequently 18 variable ones. Hence *the locus of the zero point in S_3 is a curve of order* 18 *and will be denoted by* C^{18}.

It has already been pointed out that the order of the group of the y's is double the order of the group of the z's and that to express a form in y and z covariantly its points and planes in S_3 must appear squared. This C^{18} can evidently be represented covariantly if we take the 10 quadratic combinations of the y's from the equations (50) from which we can eliminate the factor $t_1t_2t_4$ and thereby eliminate the fixed intersections each taken twice, and if we take in primed variables the corresponding quadratic combinations of the U's as the coefficients of these quadratic combinations of the y's. This form is

$$f(t'^3, t^9) = 0, \tag{51}$$

and is of the third order in t', and of the ninth order in t. The number of variable points in which this nonic intersects K is 36, which is double the order of C^{18}, since its points appear squared in (51). If $t = t'$ in (51) we find a form of order 12 which is $K^3 + 16H^2$, where H is the Hessian of K. We can then say that the form (51) is the third polar of $K^3 + 16H^2$ plus covariant terms containing the line coördinates tt'. To obtain these further terms one would make use of the complete system of invariants and covariants of K which has been calculated and tabulated by Gordan.*

4. Summary. The results obtained may be briefly summarized. The well known elliptic modular functions associated with the elliptic norm curve E^7 and the algebraic relations connecting them have been readily found from the geometric point of view. The system of contact cubics in (37), the coördinates of the modular lines L_m and L_m' and the parametric representation of the locus of the zero point in S_3 are *new* types of functions. The system of modular functions $B\gamma$ (in Klein's notation)† which define a curve of order 14 has not been found.

If a pair of points in the involution on the curve E^7 is isolated, the quadratic irrationality associated with the curve E^7 is obtained. This irrationality can be obtained from the system y_i in (44). If we substitute the values of these y_i in any of the quadrics (19) (except those bilinear in y and z), ρ is obtained as the square root of the reciprocal of a conic $g(t^6, z^2)$. This conic has the form

$$g(t^6, z^2) = z_1^2(2t_1t_2^4t_4 + t_1^2t_2t_4^3) + z_2^2(2t_1t_2t_4^4) + z_4^2(t_1^4t_2t_4) + z_1z_2(t_1^2t_2^3t_4 - 2t_1^3t_4^3) + z_2z_4(-4t_1^3t_2^3 - t_2^5t_4) - z_1z_4(3t_2^3t_4^3), \tag{52}$$

* Mathematische Annalen, vol. 17 (1880), pp. 217, 359.

† See K. F., vol. 2, p. 396–397.

and constitutes the part in z of a quadric on the curve E^7 whose part in y is the square of the half period plane (47).* We can now write down the parametric representation of the curve E^7. It is

$$
\begin{aligned}
y_0' &= +\,2t_1^2t_2^3z_2\ (41) - (2t_1^2t_2^2t_4z_2 + 2t_1t_2^3t_4z_1)\ (12),\\
y_1' &= -\,t_1t_2^3t_4z_4\ (12) + (t_1t_2^3t_4z_2 - t_1^3t_2^2z_1)\ (41),\\
y_2' &= -\,t_2t_4^3t_1z_1\ (24) + (t_2t_4^3t_1z_4 - t_2^3t_4^2z_2)\ (12),\\
y_4' &= -\,t_4t_1^3t_2z_2\ (41) + (t_4t_1^3t_2z_1 - t_4^3t_1^2z_4)\ (24),\\
z_1' &= z_1t_1t_2t_4\sqrt{g(t^6, z^2)},\\
z_2' &= z_2t_1t_2t_4\sqrt{g(t^6, z^2},\\
z_4' &= z_4t_1t_2t_4\sqrt{g(t^6, z^2)}.
\end{aligned}
\tag{53}
$$

If t is on K the above system maps the doubled C^2 upon the E^7. It should be noted that the y's vanish for $z_i = t_i$ and the z's vanish when z is on a half period point.†

* Professor Sharpe of Cornell pointed out this fact to me, as well as a method of eliminating an extraneous factor $t_1t_2t_4$ from the parametric representation of the curve E^7. I append the method in a foot note at the end of the paper.

† All the terms in y_i' contain the factor $t_1t_2t_4$ except one term in y_0' and this term contains a z_4. If we now find the intersection of the pencil of lines through the point t, $\lambda_1(42) + \lambda_2(14) = 0$ and the polar conic C^2, we get the following values for z_i:

$$
\begin{aligned}
z_1 &= \lambda_1^2(-2t_1^2t_2 - t_4^3) + \lambda_2^2t_1t_2t_4 - 3t_2^2 + t_4\lambda_1\lambda_2,\\
z_2 &= \lambda_1^2t_1t_2^2 + \lambda_2^2(-2t_2^2t_4 - t_1^3) - (2t_1^2t_2 + t_4^3)\lambda_1\lambda_2,\\
z_4 &= \lambda_1^2t_1t_2t_4 + \lambda_2^2t_2t_4^2\ t_1^2 + t_4\lambda_1\lambda_2.
\end{aligned}
$$

Hence when these values are put in (53) the factor $t_1t_2t_4$ can be removed.

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LINEAR EQUATIONS WITH TWO PARAMETERS*

BY

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The problem of the existence of solutions of two linear self-adjoint differential equations of the second order with two parameters has been treated by Hilbert, † Richardson,‡ and Yoshikawa.§ There are corresponding problems in the theory of linear equations in infinitely many unknowns, and in the theory of linear integral equations. Part I of this paper deals with the first problem, the existence of characteristic numbers λ and μ for the system of equations

$$(1)\qquad \begin{aligned} u_i &= \lambda \sum_{j=1}^{\infty} k_{ij}\, u_j + \mu \sum_{j=1}^{\infty} l_{ij}\, u_j \qquad (i=1,2,\ldots), \\ v_k &= \lambda \sum_{l=1}^{\infty} m_{kl}\, v_l - \mu \sum_{l=1}^{\infty} n_{kl}\, v_l \qquad (k=1,2,\ldots), \end{aligned}$$

where the matrices are subject to certain conditions (§1), and the expansion of the determinant matrix in terms of the solutions. Part II deals with the second problem, the existence of characteristic numbers λ and μ for the equations

$$u(x) = \lambda \int_a^b K(x,y)\, u(y)\, dy + \mu \int_a^b L(x,y)\, u(y)\, dy,$$

$$v(s) = \lambda \int_c^d M(s,t)\, v(t)\, dt - \mu \int_c^d N(s,t)\, v(t)\, dt,$$

where the kernels are subject to certain conditions (§3), and the expansion of arbitrary functions of two variables in terms of the characteristic functions.

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* Presented to the Society, October 25, 1919.

† *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, pp. 262–267.

‡ *Theorems of oscillation for two linear differential equations of the second order with two parameters*, these Transactions, vol. 13 (1912), pp. 22–34.

§ *Ein zweiparametriges Oscillationstheorem*, Göttinger Nachrichten, 1910, pp. 586–594.

I. Linear equations in infinitely many unknowns

1. **Existence of solutions.** Consider the system of linear equations (1) where the matrices K, L, M, N are real and symmetric, the sum of the squares of the elements in each matrix is convergent, the two matrices L and N are positive definite and $k_{ij}n_{kl} + l_{ij}m_{kl} \not\equiv 0$. We shall show that under these conditions there exist values of λ and μ for which the equations (1) have solutions of finite norm and not identically zero, that is, neither $\{u_i\}$ nor $\{v_k\}$ is identically zero. The method consists of the elimination of μ from (1), the transformation of the resulting system of equations in the one parameter λ into a system of the form (9) for which the existence of characteristic numbers λ_i has already been established, the substitution of these λ_i in one system of equations of (1), and the proof of the existence of corresponding characteristic numbers $\mu_{i\alpha}$, and finally the proof that λ_i and $\mu_{i\alpha}$ are also characteristic numbers for the other equations of (1).

The elimination of μ in (1) gives the system

$$u_i \sum_l n_{kl}v_l + v_k \sum_j l_{ij}u_j = \lambda\left(\sum_j k_{ij}u_j \sum_l n_{kl}v_l + \sum_j l_{ij}u_j \sum_l m_{kl}v_l\right). \tag{2}$$

Since L and N are positive definite their characteristic numbers are positive; call them α_i^2 and β_k^2. The matrices formed from the corresponding solutions l_{ij}^* and n_{kl}^* are orthogonal,* and hence the system (2) is equivalent to

$$u_i^* v_k^* \left(\frac{1}{\alpha_i^2} + \frac{1}{\beta_k^2}\right) = \lambda \sum_{j,l}\left(k_{ij}^* \frac{\delta_{kl}}{\beta_k^2} + m_{kl}^* \frac{\delta_{ij}}{\alpha_i^2}\right) u_j^* v_l^*, \tag{3}$$

where

$$\begin{aligned} &u_i^* = \sum_j l_{ij}^* u_j \quad , \quad v_k^* = \sum_l n_{kl}^* v_l, \\ &K^* = L^* K L^{*\prime} \quad , \quad M^* = N^* M N^{*\prime}, \\ &\delta_{ij} = 1 \text{ if } i = j, \text{ and } = 0, \text{ if } i \neq j. \end{aligned} \tag{4}$$

With the substitution

$$\begin{aligned} x_\xi^* &= u_i^* v_k^*, \\ a_{\xi\eta} &= k_{ij}^* \frac{\delta_{kl}}{\beta_k^2} + m_{kl}^* \frac{\delta_{ij}}{\alpha_i^2}, \\ c_\xi^2 &= \frac{1}{\alpha_i^2} + \frac{1}{\beta_k^2}, \end{aligned} \tag{5}$$

* For notation and terminology see Hilbert, loc. cit., Kap. XI, and Hellinger, E., *Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen*, Journal für Mathematik, vol. 136, pp. 210–262.

the system (3) becomes

$$c_\xi^2 x_\xi^* = \lambda \sum_\eta a_{\xi\eta} x_\eta^*. \tag{6}$$

The matrix A is symmetric, for the interchange of i with j and of k with l, or the interchange of ξ with η, leaves A unchanged. The matrix

$$\frac{a_{\xi\eta}}{c_\xi^2} = k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} \delta_{kl} + m_{kl}^* \frac{\beta_k^2}{\alpha_i^2 + \beta_k^2} \delta_{ij}$$

is completely continuous. Let $x_\xi = x_{ik}$ be of norm $\leqq 1$. Since

$$\frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} < 1,$$

$$\left| \sum_{i,j,k} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} \right| \leqq \sqrt{\sum_{i,j} k_{ij}^{*2}} \sum_{i,k} x_{ik}^2$$

and

$$\left| \sum_{i,j,k=1}^{\infty} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} - \sum_{i,j,k=1}^{n} x_{ik} k_{ij}^* \frac{\alpha_i^2}{\alpha_i^2 + \beta_k^2} x_{jk} \right| \leqq \sqrt{\sum_{i,j=n+1}^{\infty} k_{ij}^{*2}}.$$

By hypothesis $\Sigma_{ij} k_{ij}^{*2}$ converges, and $\lim_{n \to \infty} \Sigma_{i,j=n}^{\infty} k_{ij}^{*2} = 0$. Hence the matrix $k_{ij}^* \delta_{kl} \alpha_i^2/(\alpha_i^2 + \beta_k^2)$ is completely continuous, and similarly for $m_{kl}^* \delta_{ij} \beta_k^2/(\alpha_i^2 + \beta_k^2)$. This shows that $a_{\xi\eta}/c_\xi^2$ is completely continuous. From the inequality

$$\left| \frac{a_{\xi\eta}}{c_\xi c_\eta} \right| \leqq \left| a_{\xi\eta} \right| \left(\frac{1}{c_\xi^2} + \frac{1}{c_\eta^2} \right)$$

it follows that $a_{\xi\eta}/c_\xi c_\eta$ is completely continuous. The matrix $a_{\xi\eta}/c_\xi c_\eta$ is symmetric, and not identically zero since $KN + LM \neq 0$, hence there exist* values of λ for which the system

$$y_\xi = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\xi c_\eta} y_\eta \tag{7}$$

has solutions of finite norm. From this it follows that the system

$$z_\xi = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\eta^2} z_\eta \tag{8}$$

* Hilbert, loc. cit., p. 148.

has solutions $\{z_\xi = y_\xi c_\xi\}$ of finite norm for the same value of λ. Since the matrix $a_{\xi\eta}/c_\xi^2$ is completely continuous the adjoint* system

$$x_\xi^* = \lambda \sum_\eta \frac{a_{\xi\eta}}{c_\xi^2} x_\eta^* \tag{9}$$

also has solutions $\{x_\xi^*\}$ of finite norm. For each value of λ there is only a finite number of linearly independent solutions of (8) and an equal number of linearly independent solutions of (9), and the relation between the solutions of (8) and (9) is

$$z^\xi = c_\xi^2 x_\xi^*. \tag{10}$$

A solution $\{x_\xi^*\}$ of (9) leads to a solution $\{x_{ik}\}$, such that $\Sigma_{i,k} x_{ik}^2$ is convergent, of the system

$$\sum_j l_{ij}x_{jk} + \sum_l n_{kl}x_{il} = \lambda \sum_{j,l} (k_{ij}x_{jl}n_{kl} + l_{ij}x_{jl}m_{kl}). \tag{11}$$

We shall show later that x_{ik} either has the form $u_i v_k$, or is a sum of such terms.

Return to the system (1), and let λ be a value for which the system (11) has a solution x_{ik}. Assume first that u_i^* is the only solution of

$$u_i^* = \lambda \sum_j k_{ij}u_j^* \tag{12}$$

with the understanding that $\{u_i^*\}$ may be $= 0$, and that $\Sigma_i u_i^{*2} = 1$ if u_i^* is not $= 0$. There exists a limited matrix† P such that

$$P = \lambda K + \lambda PK - U^*U^{*\prime},$$

where

$$u_{ik}^* = u_i^* \text{ if } k=1,\ u_{ik}^* = 0 \text{ if } k \neq 1.$$

If the system

$$U(x) = \lambda KU(x) + \mu LU(x) \tag{13}$$

has a solution $U(x)$ besides $U^*(x)$ for $\mu = 0$, then $U(x)$ satisfies

$$\begin{gathered} U(x) = \mu\{LU(x) + PLU(x)\} + cU^*(x), \\ (U^*, LU) = 0. \end{gathered} \tag{14}$$

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* Hilbert, loc. cit., p. 165.

† Hurwitz, W. A., *On the pseudo-resolvent to the kernel of an integral equation*, these Transactions, vol. 13 (1912), pp. 405–418.

Also if (14) has a solution $U(x)$, this solution satisfies (13), and the two equations (13) and (14) have the same solutions except for $U^*(x)$. The two equations of (14) may be expressed by one equation, for from the second part it follows that

$$c(U^*, LU^*) = -\mu[(U^*, LLU) + (U^*, LPLU)],$$

and the substitution of this value for c gives

$$(15) \qquad U(x) = \mu\left[LU(x) + PLU(x) - \frac{U^*U^{*\prime}LLU(x) + U^*U^{*\prime}LPLU(x)}{(U^*, LU^*)}\right] = \mu QU(x).$$

The equations (14) and (15) have the same solutions except for $U^*(x)$. To establish the existence of characteristic numbers of (15), consider the system

$$(16) \qquad U(x) = \mu\left\{QU(x) - \frac{LU^*U^{*\prime}LU(x) + PLU^*U^{*\prime}LU(x)}{(U^*, LU^*)}\right\} = \mu RU(x).$$

All the solutions of (15) satisfy (16), and any solution $U(x)$ of (16) which is such that

$$(17) \qquad (U^*, LU) = 0$$

satisfies (15). This condition is satisfied by all the solutions of (16) except possibly those corresponding to the value μ_0 of μ given by

$$\mu_0[(U^*, LLU^*) + (U^*, LPLU^*)] + (U^*, LU^*) = 0.$$

A solution of (16) corresponding to μ_0 is $U^*(x)$, the solution of (13) for $\mu = 0$. By adding multiples of $U^*(x)$ any other solution of (16) corresponding to μ_0 can be made to satisfy (17), and hence we may assume that the systems (13) and (16) have the same solutions. The matrix R in (16) is such that the product of R by the symmetric positive definite matrix L is symmetric, and therefore* characteristic numbers μ_α exist and are real, $\{LU_\alpha(x)\}$ is the adjoint system of the system of solutions $U_\alpha(x)$ and the matrix R may be expressed in terms of the solutions in the following form:

$$(18) \qquad R'LF(x) = \sum_\alpha \frac{LU_\alpha(x)(LF, U_\alpha)}{\mu_\alpha},$$

* A. J. Pell, *Linear equations with unsymmetric systems of coefficients*, these Transactions, vol. 20 (1919), pp. 23–39.

where $F(x)$ is any limited linear form. From this expansion it follows not only that (16) and (13) have solutions besides $U^*(x)$, but also that the system of all the solutions $U_\alpha(x)$ of the equations

$$U_\alpha(x)=\lambda KU_\alpha(x)+\mu_\alpha LU_\alpha(x) \tag{19}$$

is such that the system $\{LU_\alpha(x)\}$ is complete. Suppose that $F(x)$ is such that

$$(F, LU_\alpha)=0;$$

then from (16) and the expansion for R the linear form

$$G(x)=LF(x)+PLF(x)$$

is such that

$$LG(x)-\frac{LU^*(x)(G, LU^*)}{(U^*, LU^*)}=0.$$

The only $G(x)$ which satisfies this is $G(x)=kU^*(x)$. From the definition of the matrix P it follows that if

$$LF(x)+PLF(x)=kU^*(x),$$

then $LF(x)=cU^*(x)$, and therefore $F(x)=0$. Hence the system $\{LU_\alpha(x)\}$ is complete.

If the system (12) had more than one linearly independent solution, similar considerations would show that again the system $\{LU_\alpha(x)\}$ is a complete system.

From (11) we obtain solutions $V_\alpha(x)$ of the second system of equations of (1) by multiplying by U_α,

$$V_\alpha(x)=\lambda MV_\alpha(x)-\mu_\alpha NV_\alpha(x), \tag{20}$$

for the characteristic numbers λ and μ_α. The solutions $V_\alpha(x)$ are given by

$$V_\alpha(x)=X'LU_\alpha(x),$$

and since $\{LU_\alpha(x)\}$ is complete the $V_\alpha(x)$ are not all zero, and the equations (19) and (20) have solutions not identically zero. We have established the following theorem.

THEOREM 1. *If the matrices K, L, M, and N are symmetric matrices such that the sum of the squares of the elements in each is convergent, if L and N are positive definite, and if $k_{ij}n_{kl}+l_{ij}m_{kl}\not\equiv 0$, there exist real values of λ and μ for which the system* (1) *has solutions u_i, v_k of finite norm and not identically zero.*

2. **Expansion of the determinant matrix.** Since every solution $u_{\alpha i}$, $v_{\alpha k}$ of finite norm of (19) and (20) satisfies (9), only a finite number of $v_{\alpha k}$ can be different from zero. It has been shown that $\{LU_\alpha(x)\}$ is a complete system, and therefore

$$x_{ik} = \sum_{\alpha=1}^{n} u_{\alpha i} v_{\alpha k},$$

for otherwise the difference between the left and right hand sides would be orthogonal to $LU_\alpha(x)$. We may therefore assume that the solutions of (11) are in the form of products $u_{\alpha i} v_{\alpha k}$.

From the expansion of the symmetric completely continuous matrix* $(a_{\xi\eta}/c_\xi c_\eta)$,

$$\frac{a_{\xi\eta}}{c_\xi c_\eta} = \sum_\alpha \frac{y_{\alpha\xi} y_{\alpha\eta}}{\lambda_\alpha},$$

it follows that

$$a_{\xi\eta} = \sum_\alpha \frac{c_\xi^2 x^*_{\alpha\xi} c_\eta^2 x^*_{\alpha\eta}}{\lambda_\alpha},$$

and this gives the following expansion for the determinant matrix:

$$k_{ij} n_{kl} + l_{ij} m_{kl} = \sum_\alpha \frac{w_{\alpha ik} w_{\alpha jl}}{\lambda_\alpha}, \tag{21}$$

where

$$w_{\alpha ik} = v_{\alpha k} \sum_j l_{ij} u_{\alpha j} + u_{\alpha i} \sum_l n_{kl} v_{\alpha l}.$$

The following theorem has been established.

THEOREM 2. *Under the conditions of Theorem* 1, *the determinant matrix* $KN + LM$ *may be expressed in terms of the solutions of* (1) *in the form* (21).

II. TWO LINEAR INTEGRAL EQUATIONS WITH TWO PARAMETERS

3. **Existence of solutions of two integral equations.** In the two linear integral equations

$$\begin{aligned} u(x) &= \lambda \int_a^b K(x,y) u(y)\,dy + \mu \int_a^b L(x,y) u(y)\,dy, \\ v(s) &= \lambda \int_c^d M(s,t) v(t)\,dt - \mu \int_c^d N(s,t) v(t)\,dt, \end{aligned} \tag{22}$$

* Hilbert, loc. cit., p. 148.

let the kernels K, L be real symmetric functions continuous in the real variables x and y for $a \leqq x \leqq b$ and $a \leqq y \leqq b$, M, N real, symmetric and continuous in the real variables s and t for $c \leqq s \leqq d$, $c \leqq t \leqq d$, $KN + LM \not\equiv 0$, and L and N positive definite in the sense that there exist no functions $f(x)$, $g(s)$ not identically zero which together with their squares are integrable in the sense of Lebesgue on (a, b) and (c, d) respectively, and such that $\int\int f(x)L(x,y)f(y)\,dxdy \leqq 0$, $\int\int g(s)N(s,t)g(t)dsdt \leqq 0$. Let $\{\varphi_i(x)\}$ be a closed orthogonal system of continuous functions for the interval (a, b), and $\{\psi_k(s)\}$ for the interval (c, d). Multiply the equations (22) by $\varphi_i(x)$ and $\psi_k(s)$, respectively, integrate, and then expand the right hand sides;

$$\int \varphi_i u = \lambda \sum_j \int\int \varphi_i K \varphi_j \int \varphi_j u + \mu \sum_j \int\int \varphi_i L \varphi_j \int \varphi_j u,$$

(23)

$$\int \psi_k v = \lambda \sum_l \int\int \psi_k M \psi_l \int \psi_l v - \mu \sum_l \int\int \psi_k N \psi_l \int \psi_l v.$$

If the equations (22) have continuous solutions $u(x)$, $v(s)$, then the system (23) has solutions of finite norm. The matrices in (23) satisfy all the conditions imposed on the matrices in Theorem 1, and therefore there exist values of λ and μ, necessarily real, for which the system (23) has solutions x_i, y_k of finite norm. In the usual way it can be shown that the functions

$$u(x) = \lambda \sum_i x_i \int K\varphi_i + \mu \sum_i x_i \int L\varphi_i,$$

$$v(s) = \lambda \sum_k y_k \int M\varphi_k - \mu \sum_k y_k \int N\psi_k$$

are continuous in (a, b) and (c, d), respectively, and satisfy the equations (22).

THEOREM 3. *If the kernels K, L, M, N are continuous real symmetric functions, L and N positive definite, and $KN + LM \not\equiv 0$, there exist values of λ and μ necessarily real and for which the equations* (22) *have continuous solutions $u(x)$, $v(s)$, not identically zero.*

4. **Properties of the solutions.** Let $u_i(x)$, $v_i(s)$ be solutions of (22) corresponding to the parameter values λ_i, μ_i, and $u_k(x)$, $v_k(s)$ to λ_k, μ_k. Since the kernels are symmetric it follows from (22) that

$$\begin{aligned}\int u_i u_k &= \lambda_i \int\int u_i K u_k + \mu_i \int\int u_i L u_k \\ &= \lambda_k \int\int u_i K u_k + \mu_k \int\int u_i L u_k,\end{aligned}$$

(24)

$$\begin{aligned}\int v_i v_k &= \lambda_i \int\int v_i M v_k - \mu_i \int\int v_i N v_k \\ &= \lambda_k \int\int v_i M v_k - \mu_k \int\int v_i N v_k.\end{aligned}$$

By subtraction we obtain

$$(\lambda_i - \lambda_k) \int\int u_i K u_k + (\mu_i - \mu_k) \int\int u_i L u_k = 0,$$

(25)

$$(\lambda_i - \lambda_k) \int\int v_i M v_k - (\mu_i - \mu_k) \int\int v_i N v_k = 0.$$

If the two parameter sets are not identical the determinant of the coefficients must equal zero:

$$\begin{vmatrix} \int\int u_i K u_k & \int\int u_i L u_k \\ \int\int v_i M v_k & -\int\int v_i N v_k \end{vmatrix} = 0. \tag{26}$$

This relation together with (24) gives the result that if (u_i, v_i), (u_k, v_k) belong to different parameter sets, the matrix

$$\begin{pmatrix} \int u_i u_k & \int\int u_i K u_k & \int\int u_i L u_k \\ \int v_i v_k & \int\int v_i M v_k & -\int\int v_i N v_k \end{pmatrix} \tag{27}$$

is of rank less than 2. If the solutions belong to the same values of the parameters, linear combinations may be formed so that this condition is satisfied.

Since the kernels L and N are positive definite the solutions u_i, v_i may be multiplied by constants such that

$$\int u_i^2 \int\int v_i N v_i + \int v_i^2 \int\int u_i L u_i = 1.$$

In the following we shall assume that the solutions have been so normalized.

Consider any set of functions $u_i(x)$, $v_i(s)$ which are continuous on (a, b), (c, d), respectively, and have the property that

$$\int u_i u_k \int\int v_i N v_k + \int v_i v_k \int\int u_i L u_k = \delta_{ik}. \tag{28}$$

If the series

$$f(x, s) = \sum_i A_i u_i(x) v_i(s) \tag{29}$$

is uniformly convergent the coefficients A_i may be determined in terms of $f(x, s)$. Multiply the equality (29) by

$$v_k(s) \int L(x, y) u_k(y) dy + u_k(x) \int N(s, t) v_k(t) dt$$

and integrate; then on account of (28)

$$A_k = \int\int\int u_k(x) L(x,y) f(y,s) v_k(s) dx dy ds + \int\int\int u_k(x) f(x,s) N(s,t) u_k(t) dx ds\, dt.$$

Similarly if the series

$$f(x, s) = \sum_i B_i [v_i(s) \int u_i(y) L(y,x) dy + u_i(x) \int N(s,t) v_i(t) dt]$$

is uniformly convergent,

$$B_i = \int\int u_i(x) f(x,s) v_i(s) dx ds.$$

Let $f(x, s)$ be any function continuous in x on (a, b) and in s on (c, d), and let

$$f_i = \int\int\int u_i(x) L(x,y) f(y,s) v_i(s) dx dy ds + \int\int\int u_i(x) f(x,s) N(s,t) v_i(t) dx ds dt.$$

Call f_i the Fourier coefficients of $f(x, s)$ with respect to u_i, v_i. If (29) is uniformly convergent,

$$\sum_i f_i g_i = \int\int\int f(x,s) L(x,y) g(y,s) dx dy ds + \int\int\int g(x,s) N(s,t) f(x,t) dx ds dt,$$

where $g(x, s)$ is a continuous function.

The Fourier coefficients of any continuous function $f(x, s)$ with respect to a set of functions u_i, v_i with the property (28) are of finite norm. For

$$(30)\quad \int\int\int F(x,s)L(x,y)F(y,s)dxdyds + \int\int\int F(x,s)N(s,t)F(x,t)dxdsdt \geqq 0,$$

and if

$$F(x,s) = f(x,s) - \sum_{i=1}^{n} f_i u_i(x) v_i(s),$$

the inequality (30) reduces to

$$\int\int\int f(x,s)L(x,y)f(y,s)dxdyds + \int\int\int f(x,s)N(s,t)f(x,t)dxdsdt$$

$$- 2\sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{n} f_i^2 \geqq 0.$$

Therefore

$$(31)\quad \sum_i f_i^2 \leqq \int\int\int f(x,s)L(x,y)f(y,s)dxdyds + \int\int\int f(x,s)N(s,t)f(x,t)dxdsdt,$$

which shows that the sequence $\{f_i\}$ is of finite norm.

5. **Expansion of arbitrary functions of two variables.** A consequence of (23) and (21) is

$$(32)\quad \int\int\int\int [K(x,y)f(x,s)N(s,t)g(y,t) + L(x,y)f(x,s)M(s,t)g(y,t)]dxdydsdt$$

$$= \sum_\alpha \frac{f_\alpha g_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ and $g(x, s)$ are any two continuous functions, and therefore if the series on the right is uniformly convergent,

$$K(x,y)N(s,t) + L(x,y)M(s,t) = \sum_\alpha \frac{W_\alpha(x,s)W_\alpha(y,t)}{\lambda_\alpha},$$

where

$$W_\alpha(x,s) = v_\alpha(s)\int L(x,y)u_\alpha(y)dy + u_\alpha(x)\int N(s,t)v_\alpha(t)dt.$$

To obtain the expansion of arbitrary functions, we first show that

$$\left\{\frac{W_\alpha(x,s)}{\lambda_\alpha}\right\}$$

is of finite norm; this would follow immediately if there were a continuous function $F(x, y, s, t)$ such that

$$\int\int\int\int F(x,y,s,t)\varphi_i(x)\varphi_j(y)\psi_k(s)\psi_l(t)dxdydsdt = b_{ikjl} = a_{ikjl}\frac{\alpha_j^2\beta_l^2}{\alpha_j^2+\beta_l^2},$$

where

$$a_{ijkl} = \int\int K(x,y)\varphi_i(x)\varphi_j(y)dxdy\,\frac{\delta_{kl}}{\beta_k^2} + \int\int M(s,t)\psi_k(s)\psi_l(t)dsdt\,\frac{\delta_{ij}}{\alpha_i^2},$$

and $\varphi_i(x)$ are the characteristic functions of $L(x,y)$ corresponding to the characteristic numbers α_i^2, and $\psi_k(s)$ the characteristic functions of $N(s, t)$ corresponding to the characteristic numbers β_k^2, for then

$$W_\alpha(x,s) = \lambda_\alpha\int\int F(x,y,s,t)W_\alpha(y,t)dydt,$$

and (31) would give the convergence of

$$\sum_\alpha \frac{W_\alpha^2(x,s)}{\lambda_\alpha^2}.$$

Denote by $F[f(x, s)]$ the transformation defined by

$$\int\int F[f(x,s)]\varphi_i(x)\psi_k(s)dxds = \sum_{j,l} b_{ikjl}\int\int f(x,s)\varphi_j(x)\psi_l(s)dxds.$$

If the function $F(x, y, s, t)$ above exists, then

$$F[f(x,s)] = \int\int F(x,y,s,t)f(y,t)dydt.$$

The transformed function of $g(s)\int L(x, y)\ f(y)dy$ exists and is a continuous function, for it may be expressed by

$$F[g(s)\int L(x,y)f(y)dy] = \sum_{i,k}\frac{\int K(x,y)\varphi_i(y)dy\int f\varphi_i}{\alpha_i^2}\,\frac{\alpha_i^2}{\alpha_i^2+\beta_k^2}\psi_k(s)\int g\psi_k$$

$$+\sum_{i,k}\frac{\varphi_i(x)\int f\varphi_i}{\alpha_i^2}\left(\frac{\beta_k^2}{\alpha_i^2+\beta_k^2}\int g\psi_k\int M(s,t)\psi_k(t)dt\right),$$

a series with continuous terms and such that the series formed from the absolute values of the terms is uniformly convergent.* Similarly the transformed func-

* Mercer, J., *Functions of positive and negative type*, Transactions of the Royal Society, A, vol. 209 (1909), pp. 415–446.

tion of $f(x)\int N(s, t)\ g(t)\ dt$ exists and is continuous. From the previous work it follows that the transformed function of $W_\alpha(x, s)$ is given by

$$F[W_\alpha(x,s)]=\frac{W_\alpha(x,s)}{\lambda_\alpha}. \tag{33}$$

From the series which represents $F[g(s)\int Lf]$ the following inequality is obtained:

$$2|F[g\int Lf]| \leqq \sum_{i,k}\frac{\left(\int f\varphi_i\right)^2}{\alpha_i^2}\left(\int g\psi_k\right)^2 + \Phi^2(x,s),$$

and hence

$$2|F[g\int Lf]| \leqq \int fLf\int g^2+\Phi^2(x,s), \tag{34}$$

where

$$\Phi^2(x,s)=2\sum_{i,k}\left(\int K(x,y)\varphi_i(y)dy\right)^2\frac{\psi_k^2(s)}{\alpha_i^2+\beta_k^2}+\frac{\varphi_i^2(x)}{\alpha_i^2}\left(\int M(s,t)\psi_k(t)dt\right)^2$$

and is a continuous function in x and s. Similarly it may be shown that

$$2|F[f\int Ng]| \leqq \int gNg\int f^2+\Psi^2(x,s), \tag{35}$$

where $\Psi^2(x, s)$ is a continuous function in x and s. From the two inequalities (34) and (35) it follows that

$$\begin{aligned}&\Phi^2(x,s)-2F[\int L(x,y)f(y,s)dy]+\int\int\int f(x,s)L(x,y)f(y,s)dxdyds\\ &+\Psi^2(x,s)-2F[\int N(s,t)f(x,t)dt]+\int\int\int f(y,s)N(s,t)f(y,t)dydsdt\geqq 0\end{aligned} \tag{36}$$

where

$$f(x,\ s)=\sum_{j=1}^{n}F_ju_j(x)v_j(s)$$

and

$$F_j=\frac{W_j(x,s)}{\lambda_j},$$

or by virtue of (33)

$$F_j=F[v_j(s)\int L(x,y)u_j(y)dy+u_j(x)\int N(s,t)v_j(t)dt].$$

The inequality (36) reduces to

$$\Phi^2(x,s) - 2\sum_{j=1}^{n} F_j^2 + \sum_{j=1}^{n} F_j^2 + \Psi^2(x,s) \geqq 0,$$

and therefore

$$\sum_{j=1}^{n} F_j^2 = \sum_{j=1}^{n} \frac{W_j^2(x,s)}{\lambda_j^2} \leqq \Phi^2(x,s) + \Psi^2(x,s).$$

This last result and the inequality (31) give the uniform convergence of the series formed from the absolute values of the terms of the series

$$\sum_{\alpha} \frac{W_\alpha(x,s) f_\alpha}{\lambda_\alpha},$$

where $f(x, s)$ is any continuous function, and from (32) it follows that

$$h(x,s) = \int\int (K(x,y)f(y,t)N(s,t) + L(x,y)f(y,t)M(s,t))dydt$$
$$= \sum_{\alpha} \frac{W_\alpha(x,s) f_\alpha}{\lambda_\alpha} = \sum_{\alpha} W_\alpha(x,s) \int h u_\alpha v_\alpha.$$

THEOREM 4. *If the kernels K, L, M, N are continuous real symmetric functions, L and N positive definite in the sense defined in §3, and $K\,N + L\,M \not\equiv 0$, any function $h(x, s)$ which can be expressed in the form*

$$h(x,s) = \int\int (K(x,y)f(y,t)N(s,t) + L(x,y)f(y,t)M(s,t))dydt,$$

where $f(x, s)$ is any continuous function, may be expanded into the uniformly and absolutely convergent series

$$h(x,s) = \sum_{\alpha} W_\alpha(x,s) \int h u_\alpha v_\alpha.$$

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THE THEORY OF FUNCTIONS OF ONE BOOLEAN VARIABLE*

BY

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1. In the algebra of Boolean† entities as developed by Ernst Schröder‡ and others, several important theorems regarding functions of Boolean variables are used. They have recently been stated by Eugen Müller.§ The following three I shall use in this paper.

P. 1. Boole's theorem. *Every function $f(x)$ of one variable can be "developed," i. e., brought into the "normal form"*

$$f(x)=ax+b\bar{x}$$

where $a=f(U)$ and $b=f(Z)$, U and Z designating the universal and the null class, respectively.

P. 2. Proposition of mean value. *The values which a function*

$$f(x)=ax+b\bar{x}$$

*can take lie "between"** ab and $a+b$, i. e., $ab \prec f(x) \prec a+b$, for all values of x.††*

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* Presented to the Society, September 7, 1922.

† Regarding the term "Boolean" cf. the paper by H. M. Sheffer, *A set of five independent postulates for Boolean algebras*, these Transactions, vol. 14 (1913).

‡ Ernst Schröder, *Vorlesungen über die Algebra der Logik*, Leipzig, 1890–95 (3 volumes).

§ Eugen Müller, *Schröder's Abriss der Algebra der Logik*, Leipzig, 1909, 1910.

** Following Schröder and others I say that a Boolean entity c lies "between" a and b, if a is wholly contained in c and c in b (or the same conditions with a and b interchanged). Josiah Royce, following Kempe, means by the statement c lies "between" a and b, that ab is wholly contained in c and c in $a+b$; cf. Josiah Royce, *The relation of the principles of logic to the foundations of geometry*, these Transactions, vol. 6 (1905). The former usage follows the analogy between the "inclusion-relation" and the "less-than relation." The latter, however, while at first a little surprising, is really the better. It includes the former whenever a is wholly contained in b (or vice versa).

†† The symbol $\prec$, which I use to designate "inclusion" (or "implication") is due to Mrs. Christine Ladd Franklin; cf. Baldwin's *Dictionary of Philosophy*, articles "syllogism" and "proposition."

P. 3. Converse of the proposition of mean value, or Eugen Müller's* theorem. *A function $f(x)=ax+b\bar{x}$ will take every assigned value c which lies in the interval ab to $a+b$ for some value of x; $x=ac+b\bar{c}=f(c)$ is such a value. And, in general, $f(x)=c$ if $x=(ac+\bar{a}\bar{c})\,v+(bc+b\bar{c})\bar{v}$, where v may take any value whatever.*†

These theorems form an important beginning of a theory of functions of Boolean variables. But the *variation* of a function as determined by the variation of an independent variable has not yet been studied at all. This is the problem which the present paper sets itself, and of which it gives a complete solution.

The main difficulty which had to be removed before the variation of a function could be studied at all, is this: $f(x)$ takes every one of its possible values for an infinite number of values of x, by Eugen Müller's theorem. If, then, we let x vary in any prescribed manner, how can we know what the effect of such a change will be on $f(x)$? From a value x_1 the independent variable may change to an indefinite number of values without producing any change whatever in $f(x)$! The behaviour of $f(x)$ is here somewhat similar to that of periodic functions in the theory of functions of a complex variable. There we bring order into the confusion by limiting the independent variable to intervals which form "periods;" so that, within these intervals, we can say, in general: when x_1 and x_2 represent two distinct values of the independent variable, $f(x_1)$ and $f(x_2)$ are also distinct. But the case is not quite the same for functions of Boolean variables: there is no such periodic repetition of the values of $f(x)$.‡

2. The difficulty is removed here by limiting the variation of x, e. g., to the *interval from ab to $a+b$.*

D. 1. For reasons which will appear presently, I shall call the values in this interval the "*effective values of x*," and the interval itself the "*effective range.*" Of course, a range which is effective with respect to one function, $f(x)=ax+b\bar{x}$, need not be effective with respect to another. Effectiveness is therefore always meant relative to a given function.

The chief property of effective values is stated in the following fundamental proposition:

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* As far as I am aware this proposition was first stated and proved by Eugen Müller, loc. cit., pp. 82–83 and p. 111.

† Schröder's formula for the "general" solution of an equation

$$ax+\bar{b}\bar{x}=Z,$$

namely $x=\bar{a}\omega+b\bar{\omega}$ provided $ab=Z$, is a special case of this theorem. For different expressions, cf. the discussion of the inverse function in the present paper.

‡ For functions of a Boolean variable we have indeed $f(x+uab+\bar{v}\bar{a}\bar{b})=f(x)$ somewhat resembling the definition of doubly periodic functions. But u and v are here not "integer" but arbitrary variables.

P. 4. Proposition. *If x_1 and x_2 are effective values of x relative to $f(x)=ax+b\bar{x}$, $f(x_1)$ differs from $f(x_2)$ when, and only when, x_1 differs from x_2, i. e.,*

$$(ab \prec x_1 \prec a+b)\,(ab \prec x_2 \prec a+b) \prec [(x_1 \neq x_2) = \{f(x_1) \neq f(x_2)\}].$$

In other words: If we limit x to the effective range, then every change in x produces a change in $f(x)$. It is this property which justifies the name "effective range."

Proof. I show that, under the hypothesis for x_1 and x_2, $f(x_1)$ is identical with $f(x_2)$ when, and only when, x_1 is identical with x_2; from which the theorem follows by contraposition. Let

$$f(x_1) = ax_1 + b\bar{x}_1 \text{ and } f(x_2) = ax_2 + b\bar{x}_2;$$

then

$$\begin{aligned}(1)\qquad [f(x_1)=f(x_2)] &= (ax_1 + b\bar{x}_1 = ax_2 + b\bar{x}_2)\\ &= [(ax_1 + b\bar{x}_1)\,(\bar{a}x_2 + \bar{b}\bar{x}_2) + (\bar{a}x_1 + \bar{b}\bar{x}_1)\,(ax_2 + b\bar{x}_2) = Z]\\ &= [(a\bar{b} + \bar{a}b)\,(x_1\bar{x}_2 + \bar{x}_1x_2) = Z].\end{aligned}$$

But

$$(ab \prec x_1 \prec a+b) = (ab\bar{x}_1 + \bar{a}\bar{b}x_1 = Z) \prec (ab\bar{x}_1x_2 + \bar{a}\bar{b}x_1\bar{x}_2 = Z),$$

and

$$(ab \prec x_2 \prec a+b) = (ab\bar{x}_2 + \bar{a}\bar{b}x_2 = Z) \prec (abx_1\bar{x}_2 + \bar{a}\bar{b}\bar{x}_1x_2 = Z),$$

or:

$$(2)\qquad (ab \prec x_1 \prec a+b).(ab \prec x_2 \prec a+b) \prec [(ab + \bar{a}\bar{b})\,(x_1\bar{x}_2 + \bar{x}_1x_2) = Z].$$

By combining (1) and (2) we obtain:

$$(1)\cdot(2) \prec [(ab + a\bar{b} + \bar{a}b + \bar{a}\bar{b})\,(x_1\bar{x}_2 + \bar{x}_1x_2) = Z] \prec (x_1\bar{x}_2 + \bar{x}_1x_2 = Z) = (x_1 = x_2).$$

So that, under the hypotheses for x_1 and x_2,

$$[f(x_1)=f(x_2)] \prec (x_1=x_2).$$

But from the uniquely determining character of the fundamental operations it follows that

$$(x_1=x_2) \prec [f(x_1)=f(x_2)],$$

i. e., $f(x)$ is always a "uniform" function.*

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* For the term "uniform" see Forsyth, *Theory of Functions of a Complex Variable*, (1900), p. 15.

We obtain therefore:

$$[f(x_1)=f(x_2)]=(x_1=x_2)$$

provided x_1 and x_2 are effective values.

From this our theorem follows by contraposition:

$$[f(x_1)\neq f(x_2)]=(x_1\neq x_2).$$

Within the effective range $f(x)$ is therefore a uniform function which does not "repeat" its values.

D. 2. The values of x which lie either in the interval Z to ab, or in the interval Z to $\bar{a}\bar{b}$, I call "*ineffective.*" By an "*ineffective range*", I mean a collection of ineffective values.

This name is justified by the following:

P. 5. Proposition. *However x may vary, provided it belongs to the ineffective range, $f(x)$ remains constant, i. e., is not affected by the change in x.*

Proof: To prove this theorem, I show that (1) for all values of x such that $x \prec ab$ the function has the value $f(x)=b$ and (2) for all values of x such that $x \prec \bar{a}\bar{b}$ the function has also the value $f(x)=b$; i. e.,

$$[f(x)=ax+b\bar{x}]\,(x \prec ab) \prec [f(x)=b]$$

and

$$[f(x)=ax+b\bar{x}]\,(x \prec \bar{a}\bar{b}) \prec [f(x)=b].$$

(1)
$$\begin{aligned}(x \prec ab)\ &[f(x)=ax+b\bar{x}]=(x=abx)\ [f(x)=ax+b\bar{x}]\\ &\prec [f(x)=abx+b(\bar{a}+\bar{b}+\bar{x})]=[f(x)=b(ax+\bar{a}+\bar{x})]\\ &\prec [f(x)=b(x+\bar{a}+\bar{x})]=[f(x)=b];\end{aligned}$$

(2)
$$\begin{aligned}(x \prec \bar{a}\bar{b})\ &[f(x)=ax+b\bar{x}]=(x=\bar{a}\bar{b}x)\ [f(x)=ax+b\bar{x}]\\ &\prec [f(x)=b(a+b+\bar{x})] \prec [f(x)=b].\end{aligned}$$

Q. E. D.

Given, therefore, a function of one Boolean variable, it is always possible, by Boole's theorem, to put it into the "normal" form; and therefore to determine the effective range of x, as well as the ineffective range.

The following theorem leads to a still further distinction which is sometimes important.

P. 5. Proposition. *The effective range of x contains in general (namely provided $ab \neq Z$) an ineffective part, which is innocuous.*

For let the effective values of x be designated by x_e; then, as they lie "between" ab and $a+b$, they can be represented* thus;

$$\begin{aligned}x_e&=ab+v(a+b)=ab+v(ab+a\bar{b}+\bar{a}\bar{b})\\ &=ab+v\,(a\bar{b}+\bar{a}\bar{b}),\end{aligned}$$

* See Schröder, loc. cit., vol. I, p. 426.

where v may take any value between Z and U. By this representation every effective value of x is split into two parts, namely ab and $v(a\bar{b} + \bar{a}b)$. It is the ab part which is ineffective, i. e., it contributes nothing to the value of $f(x)$; or

$$f[ab + v(a\bar{b} + \bar{a}b)] = f[v(a\bar{b} + \bar{a}b)].$$

For

$$(1) \quad \begin{aligned} f[ab + v(a\bar{b} + \bar{a}b)] &= a[ab + v(a\bar{b} + a\bar{b})] + b[(\bar{a} + \bar{b})\ (\bar{v} + ab + \bar{a}\bar{b})] \\ &= ab + va\bar{b} + \bar{a}b\bar{v} = a(b + v) + \bar{a}b\bar{v} = b(a + \bar{v}) + av, \end{aligned}$$

and

$$(2) \quad \begin{aligned} f[v(a\bar{b} + \bar{a}b)] &= va\bar{b} + b(\bar{v} + ab + \bar{a}\bar{b}) = va\bar{b} + b\bar{v} + ab \\ &= a(b + v) + b\bar{v} = b(a + \bar{v}) + av. \end{aligned}$$

I call this ineffective part of x_e "*innocuous*" to indicate that it does not invalidate the fundamental proposition

$$[f(x_e') \neq f(x_e'')] = (x_e' \neq x_e'')$$

which was proved above (P. 4) for effective values of x. The reason why this ineffective part of x_e is innocuous is clear: it, *as a whole*, is part of every x_e, so that the variation of x_e does not take place in it at all.

D. 3. But this consideration leads to the *definition of the wholly-effective range of* x. By this I mean the collection of values which lie between Z and $a\bar{b} + \bar{a}b$, i. e.,

$$Z \leqslant x_\omega \leqslant a\bar{b} + \bar{a}b,$$

where x_ω designates the wholly-effective values of x. It is obvious that the wholly-effective range of x is obtained by omitting the ineffective part ab from the effective range.

We can write x_ω in the form:

$$x_\omega = v(a\bar{b} + \bar{a}b)$$

where

$$Z \leqslant v \leqslant U.$$

If we choose the regional interpretation of our Boolean entities, which in the present paper is the preferred interpretation, then the wholly-effective range is, in general, represented by a disconnected region. A value in this range can be expressed thus:

$$(x_\omega = x_1 + x_2)\ (x_1 \leqslant ab)\ (x_2 \leqslant \bar{a}b).$$

P. 7. Proposition. (1) *For every wholly-effective value x_ω there exists an effective value x_e, determined by the condition.*

$$x_e = x_\omega + ab$$

and (2) *for every effective value x_e there exists a wholly-effective value x_ω, determined by the condition*

$$x_\omega = x_e\, \bar{a}\bar{b}.$$

For

$$(1) \qquad \begin{aligned}(ab \prec x_e \prec a + b) &= [x_e = ab + v(a + b)] \\ &= [x_e = ab + v(a\bar{b} + \bar{a}b)] = [x_e = ab + x_\omega]\end{aligned}$$

and

$$(2) \qquad x_e\, \bar{a}\bar{b} = [ab + v(a + b)] \cdot \bar{a}\bar{b} = v(a\bar{b} + \bar{a}b) = x_\omega.$$

D. 4. I shall call an x_e and an x_ω thus related "*associated values.*"

For wholly-effective values we have the same theorem (P. 4) which was demonstrated for effective values, namely

P. 8. Proposition.

$$[f(x_\omega') \neq f(x_\omega'')] = (x_\omega' \neq x_\omega'')$$

where x_ω' and x_ω'' designate any two wholly-effective values.

An independent analytical proof of this theorem can be made in a manner perfectly analogous to the one given for P. 4. But to give an illustration of our new methods, I shall present the following.

Let us start with the definite value

$$f(x_\omega') = ax_\omega' + b\bar{x}_\omega' = k$$

and consider the function

$$F(x) = f(x_\omega' + x) = a(x_\omega' + x) + b(\overline{x'_\omega + x})$$

where x shall be limited in its variation to the range Z to ab, i. e.,

$$\prod_x (x \prec ab).^*$$

I show that the values of x in this range are ineffective relative to $F(x)$.

For:

$$\begin{aligned}F(x) &= F(U)x + F(Z)\bar{x} \quad \text{(by Boole's theorem)} \\ &= ax + k\bar{x}.\end{aligned}$$

* I use Schröder's notation to express: "for *all* values of x." Cf. Schröder, *Algebra der Logik*, vol. II, p. 26, et al.

The range from Z to ak is therefore an ineffective range of x relative to $F(x)$, according to D. 2.

But

$$ab < k$$

by the Proposition of Mean Value applied to $f(x_\omega')$, and

$$(ab < k) < (ab < ak),$$

so that

$$(x < ab) < (x < ak);$$

i. e., the values of x which are limited to the range Z to ab lie also in the range Z to ak and are therefore ineffective relative to $F(x)$, i. e., $F(x)$ remains constant while x varies from Z to ab. But for $x=Z$, we have

$$F(x)=f(x_\omega')=k.$$

$F(x)$ keeps therefore the value k even when x takes the value ab. But

$$x_\omega'+ ab=x_\epsilon' \text{ (by P. 7).}$$

We find therefore that, if x_ω' and x_ϵ' are two associated values, then we have always

$$f(x_\omega')=f(x_\epsilon')$$

and the present theorem is reduced to the previous one. Q.E.D.

Let us state the proposition by which this reduction is effected as a separate proposition and add an analytical proof.

P. 9. Proposition. If x_ω' and x_ϵ' are two associated values, i. e., if

$$x_\epsilon'=x_\omega' + ab$$

then

$$f(x_\epsilon'), =f(x_\omega'),$$

i. e., $f(x)$ takes the same value for an effective value of x and its associated wholly-effective value.

Proof.

$$\begin{aligned} f(x_\epsilon') &= f(x_\omega' + ab) = ax_\omega'' + ab + \bar{a}b\bar{x}_\omega' \\ &= ax_\omega' + b(a + \bar{a}\bar{x}_\omega') = ax_\omega' + b(a + \bar{x}_\omega') \\ &= ax_\omega' + b\bar{x}_\omega' + ab = ax_\omega' + b\bar{x}_\omega' = f(x_\omega'). \end{aligned}$$

Q. E. D.

3. We have shown that a function $f(x)$ is a uniform function if x is restricted to the effective (or the wholly-effective) range. I prove next

P. 10. Proposition. *$f(x)$ will take any one of its possible values, as k, for one and only one effective value of x, and*

$$x_e = f(k) = ak + b\bar{k}$$

is this value.

Proof. (1) By Eugen Müller's Proposition, $x_1 = f(k) = ak + b\bar{k}$ is a value for which $f(x_1) = k$.

(2) $f(k)$ is an effective value of x, i. e., $f(k) = x_e'$, for it lies in the effective range ab to $a + b$, by the Proposition of Mean Value.

(3) If $f(x_e') = k$, for an effective value x_e', and x_e'' is an effective value such that $x_e' \neq x_e''$, then $f(x_e') \neq f(x_e'') \neq k$ (P. 4). Q. E. D.

This proposition allows us to limit x to the effective range and yet obtain all the possible values of $f(x)$.

We may even limit x to the wholly-effective range and still obtain all the possible values of $f(x)$, as is shown by the following:

P. 11. Proposition. *A function $f(x)$ will take any one of its possible values, as k, for one (and only one) wholly-effective value of x, and $x_\omega = \bar{b}k + b\bar{k}$ is this value.*

Proof. By the previous proposition (P. 10) there is a value

$$x_e = f(k) = ak + b\bar{k}$$

such that

$$f(x_e) = k.$$

Therefore there must be a value x_ω, namely the associated wholly-effective value, such that $f(x_\omega) = f(x_e) = k$ (P. 7 and P. 9).

But $x_\omega = x_e \cdot \bar{a}\bar{b}$, (P. 7), i. e.,

$$x_\omega = (ak + b\bar{k})\bar{a}\bar{b} = a\bar{b}\bar{k} + \bar{a}b\bar{k} = a\bar{b}k + \bar{a}\bar{b}k + \bar{a}b\bar{k} + ab\bar{k} = \bar{b}\bar{k} + b\bar{k},$$

since

$$(ab \leqslant k \leqslant a + b) = (ab\bar{k} = Z)\ (\bar{a}\bar{b}k = Z),$$

and this value is the only wholly-effective value for which $f(x_\omega) = k$ (by P. 9).

Q.E.D.

4. The values which a function of one Boolean variable takes may be grouped into three ranges, in terms of the regional interpretation.

D. 5, 6, 7, 8. I say that a function

$$f(x) = ax + b\bar{x}$$

consists, in general, of three possible regions, namely

$$ab,\ a\bar{b} \text{ and } \bar{a}b$$

which I shall call the "*constant region* (ab) (D. 5), the "*increment region*" $(a\bar{b})$, (D. 6) and the "*decrement region*" $(\bar{a}b)$ (D. 7), respectively. The remaining region $\bar{a}\bar{b}$, I call the "*impossible region*" (D. 8) for $f(x)$.* This distinction is preparatory to the statement of the principal theorem (P. 16); the reasons for naming these regions as I do will presently become apparent.

Let us first consider a few special cases.

Case 1. Let x be any (effective) value which lies wholly in the *increment region*, $a\bar{b}$, *of* $f(x)$, i. e.,

$$(x \prec a\bar{b}) = (x = ma\bar{b})$$

where

$$Z \prec m \prec U.$$

Then

$$\begin{aligned} f(x) = ax + b\bar{x} &= m \cdot a\bar{b} + b(\bar{m} + \bar{a} + b) = m \cdot a\bar{b} + b \\ &= x + b. \end{aligned}$$

Or, if x is the region marked thus: $\equiv$, and $f(x)$ is the region marked thus $||||$, the accompanying Figure 1 will represent the relation of $f(x)$ to x.

In other words: as long as x remains wholly within the increment region, otherwise varying in any manner whatsoever, $f(x)$ will be obtained by adding a constant amount, namely b, to it. Or, put differently:

P. 12. Proposition. *If x flows in any manner whatsoever into the increment region $a\bar{b}$, or out of it, $f(x)$ will flow with it into $a\bar{b}$, or out of it.*

Case 2. Let x be any (effective) value which lies wholly in the *decrement region*, $\bar{a}b$, *of* $f(x)$, i. e.,

$$(x \prec \bar{a}b) = (x = n\bar{a}b)$$

where

$$Z \prec n \prec U.$$

Then

$$f(x) = ax + b\bar{x} = an\bar{a}b + b(\bar{n} + a + \bar{b}) = b\bar{x}.$$

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* The relation between the "possible" regions of $f(x)$ and the effective values of x should be noticed.

With the same method of representation the accompanying figure 2 will exhibit the relation of $f(x)$ to x in this case.

This relation is stated in the following proposition:

P. 13. **Proposition.** *If x flows, in any manner whatsoever, into the decrement region $\bar{a}b$, $f(x)$ flows out of it by the same amount; and $f(x)$ flows into the decrement region, if x flows out, and by the same amount. So that $f(x)$ will always fill the decrement region unless forced out of it by the inflow of an x of the same amount.*

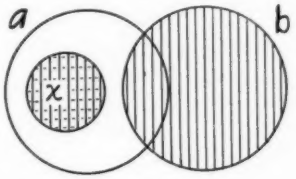

FIG. 1.

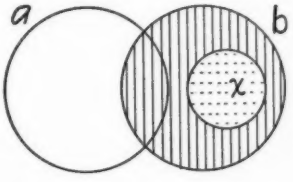

FIG. 2.

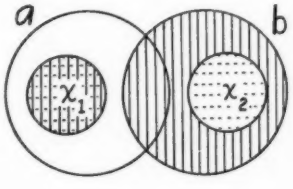

FIG. 3.

Case 3. Let x consist of two pieces x_1 and x_2 such that

$$(x=x_1+x_2)\cdot(x_1 \prec a\bar{b})\ (x_2 \prec \bar{a}b),$$

i. e.,

$$x=m\cdot a\bar{b}+n\bar{a}b,$$

where m and n take any values whatever.

Then

$$f(x)=m\cdot a\bar{b}+b\bar{x}=x_1+b\bar{x}=x_1+b\bar{x}_1\cdot\bar{x}_2=x_1\cdot+b\bar{x}_2.$$

This is represented in figure 3. In other words:

P. 14. Proposition. *The effect of the two parts of x on $f(x)$ can be obtained by considering their effects separately and combining the partial effects.*

It will be recalled that the values of x are in these three cases effective, and indeed wholly-effective, relative to $f(x)$.

Case 4. Let x be any value which lies wholly within the *constant region* ab. Its variation will have no effect on $f(x)$. Hence:

P. 15. Proposition. *The $f(x)$ always fills the constant region, by the Proposition of Mean Value, irrespective of x's flowing into or out of it. No change of x whatever can force $f(x)$ out of the constant region ab; or into the impossible region $\bar{a}\bar{b}$ (the latter also by the Proposition of Mean Value).*

We are now prepared to state in a single proposition the behavior of $f(x) = ax + b\bar{x}$ for any change of x:

P. 16. Principal Theorem. *Every change of x, or part of x, in the increment region $a\bar{b}$ produces the same change in $f(x)$; in the decrement region $\bar{a}b$ it produces the opposite effect; and a change of x in any other region has no effect whatsoever on $f(x)$, which always occupies ab, and never enters $\bar{a}\bar{b}$.*